THE INVARIANTS OF 2x2 MATRICES, THEIR ALGEBRAIC RELATIONS AND THE CORRESPONDING MODULI PROBLEM

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE KATHOLIEKE UNIVERSITEIT TE NIJMEGEN, OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. A.J.H. VENDRIK, VOLGENS BESLUIT VAN HET COLLEGE VAN DECANEN IN HET OPENBAAR TE VERDEDIGEN OP DONDERDAG 30 SEPTEMBER 1976, DES NAMIDDAGS TE 4 UUR

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KRIPS REPRO - MEPPEL

You got to learn how to fall
Before you learn to fly

(Paul Simon)

Ik bedank Willy van de Sluis-Gerritsen voor de aandacht die zij aan de vormgeving van dit proefschrift heeft besteed.

CONTENTS

Introduction and summary

Throughout this paper k is an algebraically closed field of characteristic zero. Varieties are always reduced, separated k-schemes of finite type (not necessarily irreducible) and morphisms of varieties are k-morphisms.

A reference 2.4 (or II.2.4) indicates the definition (proposition etc.) labeled as 2.4 in §2 of the same chapter (or in chapter II, respectively).

Let $M(2,k)^n$ denote the set of ordered sequences of n 2×2 matrices with coefficients in the field k. The general linear group $Gl(2,k)$ acts (from the right) on $M(2,k)^n$ by simultaneous conjugation. Denoting the orbit of $(A_1,\dots,A_n) \in M(2,k)^n$ by $cl(A_1,\dots,A_n)$, we therefore have: $cl(A_1,\dots,A_n) = \{(T^{-1}A_1T,\dots,T^{-1}A_nT) \mid T \in Gl(2,k)\}$.

What are the moduli of this action, i.e., what are the invariants which we can attach to these orbits and to what extent do they determine the orbits? The problem is readily solved in the case n = 1 by means of the Jordan canonical form (cf. also [10]). Therefore we restrict ourselves from now on to the cases $n \geq 2$. Translating the action of $Gl(2,k)$ on $M(2,k)^n$ in terms of algebraic geometry we have the following approach: $Gl(2,k)$ can be identified with the set of closed points of the affine algebraic group G = Spec S where $S = k[T_{11},T_{12},T_{21},T_{22},(T_{11}T_{22}-T_{12}T_{21})^{-1}]$. Let $R = k[\underline{X}_1,\underline{X}_2,\dots,\underline{X}_n]$, the $\underline{X}_i$ indicating the four indeterminates $X_{i;11},X_{i;12},X_{i;21}$ and $X_{i;22}$ $(1 \leq i \leq n)$. Identify $M(2,k)^n$ with the set of closed points of the affine variety $X = \mathrm{Spec}\ R$. The action of $Gl(2,k)$ on $M(2,k)^n$ can be translated in a natural way into an algebraic action $\sigma : G \times_k X \to X$ of G on X (I, §1).

The elements of R can be considered polynomial functions on $M(2,k)^n$. Let R^G denote the subalgebra of R consisting of the functions being constant on each orbit $cl(A_1,\ldots,A_n)$. In the above language: $R^G = \{x \in R \mid \sigma^*(x) = 1 \otimes x\}$. The affine algebraic group $G = \mathrm{Spec}\ S$ is reductive [10]. G acts on the affine variety $X = \mathrm{Spec}\ R$ by means of the k-algebra homomorphism $\sigma^* : R \to S \otimes_k R$. This situation is dealt with in [10] (and also in [9]) where it is proved that R^G is a k-algebra of finite type, so $Y = \mathrm{Spec}\ R^G$ is an affine variety. Let $\phi : X \to Y$ be the morphism of varieties induced by the inclusion $R^G \subset R$. Then also in [10] it is shown that (Y,ϕ) is a quotient of X by G, i.e.,

(1) ϕ is constant on the orbits of closed points of X and

(2) given a variety Z and a morphism $\psi : X \to Z$ constant on those orbits, there is a unique morphism $\theta : Y \to Z$ such that $\psi = \theta \circ \phi$.

To what extent does this solve our problem? If we can give a finite set of generators of R^G this certainly gives us the invariants we asked for. This set of generators of R^G and the corresponding invariants are given in chapter I, §1, using [11]. But can it occur that two different orbits in $M(2,k)^n$ have the same invariants? In other words: ϕ is constant on the orbits of the closed points of X, but does it "separate" different orbits? In fact, ϕ does not. For instance look at the case $n = 2$. For each $t \in k$ let x_t be the closed point of $X = \mathrm{Spec}\ k[\underline{X}_1,\underline{X}_2]$ corresponding to $\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) \in M(2,k)^2$. The orbit $G(x_1)$ of x_1 includes all the points x_t with $t \in k^*$. So, if $y = \phi(x_1) \in Y$, the fibre $\phi^{-1}(y)$

contains $\{x_t \mid t \in k^*\}$. But $\phi^{-1}(y)$ being closed contains also x_0. Now $G(x_0) = \{x_0\}$, $x_0 \notin G(x_1)$ so ϕ does not separate the orbits $G(x_0)$ and $G(x_1)$.

Define the invariant open subset X_S of X as follows: a closed point x of X is in X_S iff the orbit of x is closed and the stabilizer $S(x)$ of x has minimal dimension. Let $Y_S = \phi(X_S)$ and $\phi' : X_S \to Y_S$ the restriction of ϕ. Then in [10] it is proved that (Y_S, ϕ') is a geometric quotient of X_S by G (cf. I, §2). Especially this implies that ϕ' separates different orbits. In chapter II we study X_S. Its closed points correspond to the irreducible elements of $M(2,k)^n$ where $(A_1,\dots,A_n) \in M(2,k)^n$ is irreducible iff the corresponding linear transformations A_i of k^2 have no common eigenvector.
In II, §1 we deduce criteria for (ir)reducibility. In §2 we prove that two irreducible elements of $M(2,k)^n$ have the same invariants iff they belong to the same orbit and in §3 we determine X_S.

As an illustration we treat the cases $n = 2$ and $n = 3$ in chapter III. If $n = 2$, Y itself is smooth. Surprisingly Y_S turns out to be the smooth part of Y in the case $n = 3$.

Let Σ be the polynomial ring over k in the $m = 2n + \binom{n}{2} + \binom{n}{3}$ indeterminates X_i $(1 \le i \le n)$, Y_i $(1 \le i \le n)$, Z_{ij} $(1 \le i < j \le n)$ and W_{ijk} $(1 \le i < j < k \le n)$.
In I, §1 we define a k-algebra homomorphism $\alpha : \Sigma \to R$ such that $\alpha(\Sigma) = R^G$, using the results of C. Procesi [11]. Obviously the elements of ker α describe the algebraic relations between the invariants. We refine in our case (i.e. 2×2 matrices) Procesi's description of ker α in the sense that we explicitly give a <u>finite</u> set of generators of ker α (IV, §1 and

§2). In IV, §3 we prove that Y_S is the smooth part of Y in all the cases $n \geq 3$.

Next we examine the singular part $Y-Y_S$ of Y in chapter V ($n \geq 3$). It turns out to be isomorphic with $\mathbb{A}^n * \mathbb{A}^n$, the symmetric product of the n-dimensional affine space $\mathbb{A}^n$ with itself (V, §1). We describe the ring of coordinates of $\mathbb{A}^n * \mathbb{A}^n$ and finally we prove that the singular part of $\mathbb{A}^n * \mathbb{A}^n$, so also of $Y-Y_S$, is isomorphic with $\mathbb{A}^n$ (V, §2).

The first five chapters of this thesis can be regarded as an investigation of the ring of invariants R^G. The best way to understand the substance of chapter VI - which treats the moduli problem corresponding to the action of $Gl(2,k)$ on $M(2,k)^n$ - is reading its §1.

I want to express my gratitude to prof. Hochster who drew my attention to the work of C. Procesi.

Chapter I. The ring of invariants. Quotient and geometric quotient.

In this chapter we give the definition of the symbols we use throughout this paper. We discuss the results of C. Procesi in connection with the ring of invariants [11]. We also give the tools of moduli theory given by Mumford [10].

§1. The ring of invariants.

$M(2,k)^n$ denotes the set of ordered sequences of n 2×2 matrices with entries in the field k. $Gl(2,k)$ acts on $M(2,k)^n$ by simultaneous conjugation. The orbit of $(A_1,\dots,A_n) \in M(2,k)^n$ under this action is denoted by $cl(A_1,\dots,A_n)$, so $cl(A_1,\dots,A_n) = \{(T^{-1}A_1T,\dots,T^{-1}A_nT) \mid T \in Gl(2,k)\}$. We formalize this action as follows:

definition 1.1. $M(2,k)^n$ is the set of closed points of the affine variety $X = \mathrm{Spec}\ R$, where $R = k[\underline{X}_1,\underline{X}_2,\dots,\underline{X}_n]$, the $\underline{X}_i$ indicating the four indeterminates $X_{i;11}, X_{i;12}, X_{i;21}$ and $X_{i;22}$. We also use the symbol $\underline{X}_i$ for the element $\begin{pmatrix} X_{i;11} & X_{i;12} \\ X_{i;21} & X_{i;22} \end{pmatrix}$ of $M(2,R)$.

definition 1.2. $Gl(2,k)$ corresponds to the set of closed points of the affine algebraic group $G = \mathrm{Spec}\ S$, where $S = k[T_{11},T_{12},T_{21},T_{22},D^{-1}]$ and $D = T_{11}T_{22}-T_{12}T_{21}$. G operates algebraically on the variety $X = \mathrm{Spec}\ R$ by means of the morphism $\sigma : G\times_k X \to X$ induced by the k-algebra homomorphism $\sigma^* : R \to S \otimes_k R$ with:

$$\sigma^*(X_{i;11}) = D^{-1}(T_{11}T_{22}\otimes X_{i;11}+T_{21}T_{22}\otimes X_{i;12}-T_{11}T_{12}\otimes X_{i;21}-T_{12}T_{21}\otimes X_{i;22})$$
$$\sigma^*(X_{i;12}) = D^{-1}(T_{12}T_{22}\otimes X_{i;11}+ T_{22}^2 \otimes X_{i;12}- T_{12}^2 \otimes X_{i;21}-T_{12}T_{22}\otimes X_{i;22})$$

$$\sigma^*(X_{i;21}) = D^{-1}(-T_{11}T_{21}\otimes X_{i;11} - T_{21}^2 \otimes X_{i;12} + T_{11}^2 \otimes X_{i;21} + T_{11}T_{21}\otimes X_{i;22})$$
$$\sigma^*(X_{i;22}) = D^{-1}(-T_{12}T_{21}\otimes X_{i;11} - T_{21}T_{22}\otimes X_{i;12} + T_{11}T_{12}\otimes X_{i;21} + T_{11}T_{22}\otimes X_{i;22}).$$

Combining σ^* with the canonical isomorphism

$S \otimes_k R \overset{\sim}{\to} k[\underline{X}_1,\ldots,\underline{X}_n,T_{11},T_{12},T_{21},T_{22},D^{-1}]$ one has:

$$\underline{X}_i \mapsto D^{-1}\begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix} \underline{X}_i \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

<u>definition 1.3</u>. R^G, the ring of invariants, is defined by $R^G = \{x \in R \mid \sigma^*(x) = 1 \otimes x\}$. R^G is a subalgebra of R.

R is in a natural way isomorphic with the k-algebra of polynomial functions from $M(2,k)^n$ to k. From the definitions above it is clear that R^G in this way corresponds to the functions being constant on the orbits of the elements of $M(2,k)^n$. Now Procesi has studied this functions in [11] from which we deduce the following results.

<u>definition 1.4</u>. T_n is the formal polynomial ring over k generated by the symbols $\mathrm{Tr}\, X_{i_1}X_{i_2}\ldots X_{i_r}$ ($r \in \mathbb{N}$, $1 \leq i_p \leq n$ for all $1 \leq p \leq r$), with the convention: $\mathrm{Tr}\, X_{i_1}\ldots X_{i_r} = \mathrm{Tr}\, X_{j_1}\ldots X_{j_r}$ iff$(j_1,\ldots,j_r)$ is obtained from $(i_1,\ldots,i_r)$ by cyclic permutation.

<u>definition 1.5</u>. The k-algebra homomorphism $\beta : T_n \to R^G$, suggested already by the notation of the indeterminates of T_n, is defined by $\beta(\mathrm{Tr}\, X_{i_1}\ldots X_{i_r})(A_1,\ldots,A_n) = \mathrm{Tr}\, A_{i_1}\ldots A_{i_r}$ for all $(A_1,\ldots,A_n) \in M(2,k)^n$.

Now Procesi proves that β is surjective, using "old" invariant theory [3]. But even more interesting is his description of the kernel of β. Let $\mathfrak{Q}$ denote this kernel. The characteristic polynomial of an element of $M(2,k)$ is: $X^2-(\mathrm{Tr}\, X)X + \frac{1}{2}[(\mathrm{Tr}\, X)^2 - \mathrm{Tr}\, X^2]I_2$. After full polarization of this polynomial one gets:

$$G(X_1,X_2) = X_1X_2+X_2X_1-(\mathrm{Tr}\, X_1)X_2-(\mathrm{Tr}\, X_2)X_1+(\mathrm{Tr}\, X_1\, \mathrm{Tr}\, X_2 - \mathrm{Tr}\, X_1X_2)I_2$$

with the property: $G(A,B) = 0$ for all $A,B \in M(2,k)$. Finally he defines

$$
\begin{aligned}
F(X_1,X_2,X_3) &= \mathrm{Tr}[\,G(X_1,X_2)X_3] \\
&= \mathrm{Tr}X_1X_2X_3 + \mathrm{Tr}X_1X_3X_2 - \mathrm{Tr}X_1\,\mathrm{Tr}X_2X_3 - \mathrm{Tr}X_2\,\mathrm{Tr}X_1X_3 + \\
&\qquad - \mathrm{Tr}X_3\,\mathrm{Tr}X_1X_2 + \mathrm{Tr}X_1\,\mathrm{Tr}X_2\,\mathrm{Tr}X_3.
\end{aligned}
$$

Viewed as an element of T_n, $F(X_1,X_2,X_3)$ clearly belongs to $\mathcal{Q} = \ker \beta$. Moreover Procesi has proved that $\mathcal{Q}$ is generated by the elements $F(M_1,M_2,M_3)$, the M_i running through all possible monomials in $X_1,X_2,\ldots,X_n$.

R^G is isomorphic with ${}^{T}n/\mathcal{Q}$. $\{\mathrm{Tr}X_{i_1}\ldots X_{i_r} + \mathcal{Q} \mid r \le 3\}$ is a set of generators for the k-algebra ${}^{T}n/\mathcal{Q}$. For instance this follows from the formula:

if $r \ge 4$ and $E = X_{i_4}\ldots X_{i_r}$ then in T_n we have

$$
\begin{aligned}
2\mathrm{Tr}X_{i_1}\ldots X_{i_r} &= F(X_{i_1}X_{i_2},X_{i_3},E) - F(X_{i_3}X_{i_1},X_{i_2},E) + F(X_{i_2}X_{i_3},X_{i_1},E) + \\
&\quad + \mathrm{Tr}X_{i_3}\,\mathrm{Tr}X_{i_1}X_{i_2}E - \mathrm{Tr}X_{i_2}\,\mathrm{Tr}X_{i_3}X_{i_1}E + \mathrm{Tr}X_{i_1}\,\mathrm{Tr}X_{i_2}X_{i_3}E + \\
&\quad + \mathrm{Tr}X_{i_1}X_{i_2}\,\mathrm{Tr}X_{i_3}E - \mathrm{Tr}X_{i_1}X_{i_3}\,\mathrm{Tr}X_{i_2}E + \mathrm{Tr}X_{i_2}X_{i_3}\,\mathrm{Tr}X_{i_1}E + \\
&+ (\mathrm{Tr}X_{i_1}X_{i_2}X_{i_3} - \mathrm{Tr}X_{i_1}\,\mathrm{Tr}X_{i_2}X_{i_3} + \mathrm{Tr}X_{i_2}\,\mathrm{Tr}X_{i_1}X_{i_3} - \mathrm{Tr}X_{i_3}\,\mathrm{Tr}X_{i_1}X_{i_2})\mathrm{Tr}E.
\end{aligned}
$$

Because $F(X_i,X_j,X_k) \in \mathcal{Q}$ for all $1 \le i \le j \le k \le n$ we can restrict the above set of generators of ${}^{T}n/\mathcal{Q}$ somewhat so that ${}^{T}n/\mathcal{Q}$ is generated by $\mathrm{Tr}X_i + \mathcal{Q}$ $(1 \le i \le n)$, $\mathrm{Tr}X_i^2 + \mathcal{Q}$ $(1 \le i \le n)$, $\mathrm{Tr}X_iX_j + \mathcal{Q}$ $(1 \le i < j \le n)$ and $\mathrm{Tr}X_iX_jX_k + \mathcal{Q}$ $(1 \le i < j < k \le n)$. For technical reasons we change the generators in a way as expressed in the following definition:

<u>definition 1.6</u>. Let Σ be the polynomial ring over k in the $m = 2n+\binom{n}{2}+\binom{n}{3}$ indeterminates X_i $(1 \le i \le n)$, Y_i $(1 \le i \le n)$, Z_{ij} $(1 \le i < j \le n)$ and W_{ijk} $(1 \le i < j < k \le n)$, so:

$\Sigma = k[X_1,\ldots,X_n;Y_1,\ldots,Y_n;Z_{1,2},\ldots,Z_{n-1,n};W_{1,2,3},\ldots,W_{n-2,n-1,n}]$.

Define $\omega : \Sigma \to {}^{T}n/\mathcal{Q}$ to be the surjective k-algebra homomorphism with

$$\begin{array}{ll}
\omega(X_i) = \mathrm{Tr}X_i + \mathcal{Q} & (1 \leq i \leq n) \\
\omega(Y_i) = 2\mathrm{Tr}X_i^2 - (\mathrm{Tr}X_i)^2 + \mathcal{Q} & (1 \leq i \leq n) \\
\omega(Z_{ij}) = 2\mathrm{Tr}X_iX_j - \mathrm{Tr}X_i\mathrm{Tr}X_j + \mathcal{Q} & (1 \leq i < j \leq n) \\
\omega(W_{ijk}) = \mathrm{Tr}X_iX_jX_k - \mathrm{Tr}X_iX_kX_j + \mathcal{Q} & (1 \leq i < j < k \leq n)
\end{array}$$

We translate these concepts back to our original R and R^G.

<u>definition 1.7</u>. $\alpha : \Sigma \to R$ is the k-algebra homomorphism such that

$$\begin{array}{ll}
\alpha(X_i) = \mathrm{Tr}\underline{X}_i = X_{i;11}+X_{i;22} & (1 \leq i \leq n) \\
\alpha(Y_i) = 2\mathrm{Tr}\underline{X}_i^2 - (\mathrm{Tr}\underline{X}_i)^2 & (1 \leq i \leq n) \\
\alpha(Z_{ij}) = 2\mathrm{Tr}\underline{X}_i\underline{X}_j - \mathrm{Tr}\underline{X}_i\mathrm{Tr}\underline{X}_j & (1 \leq i < j \leq n) \\
\alpha(W_{ijk}) = \mathrm{Tr}\underline{X}_i\underline{X}_j\underline{X}_k - \mathrm{Tr}\underline{X}_i\underline{X}_k\underline{X}_j & (1 \leq i < j < k \leq n)
\end{array}$$

Now it is clear that the diagram below is commutative. Especially we have $R^G = \alpha(\Sigma)$ so $R^G \simeq \Sigma/\ker \alpha$.

1.8.

$$\begin{tikzcd}
\Sigma \arrow[r, "\alpha"] \arrow[d, "\omega"'] & R \\
{}^{T}n/\mathcal{Q} \arrow[r, "\sim"] & R^G \arrow[u, hook] \\
& T_n \arrow[ul] \arrow[u, "\beta"']
\end{tikzcd}$$

The invariants we attach to an element of $M(2,k)^n$ are worked into

<u>definition 1.9</u>. $i : M(2,k)^n \to k^m$ is the map assigning to $(A_1,\dots,A_n)$

$$\begin{array}{cccc}
(\dots \mathrm{Tr}A_i \dots ; & \dots 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 \dots ; & \dots 2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j \dots ; & \dots \mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j \dots) \\
\uparrow & \uparrow & \uparrow & \uparrow \\
1\leq i\leq n & 1\leq i\leq n & 1\leq i<j\leq n \text{ in} & 1\leq i<j<k\leq n \text{ in} \\
 & & \text{lexicographical order} & \text{lexicographical order}
\end{array}$$

Notice that $\alpha : \Sigma \to R$ induces a unique morphism $\psi : X = \mathrm{Spec}\, R \to \mathbb{A}^m$ and that i is the restriction of ψ to the set of closed points.

Once and for all we decide on the following notation: if $a = (a_1,\dots,a_r) \in k^r$ then $\tilde{a} : k[T_1,\dots,T_r] \to k$ denotes the substitution of a. With this notation we evidently have:

<u>proposition 1.10</u>. For all $A \in M(2,k)^n$ the adjacent diagram is commutative.

$$\begin{tikzcd} \Sigma \arrow[r, "\widetilde{i(A)}"] \arrow[d, "\alpha"'] & k \\ R \arrow[ur, "\tilde{A}"'] & \end{tikzcd}$$

At this place I want to discuss the role which plays the characteristic of the field k in this thesis. Everything holds true if char $k \neq 2$ except perhaps that $\beta : T_n \to R^G$ (1.5) is not surjective. Possibly the results of Doubilet, Rota and Stein [4] concerning "characteristic-free" invariant theory can be invoked to prove the surjectivity of β.

In §2 we shall give the results of moduli theory in which the ring of invariants R^G plays an important role. Before starting this section we remark that our algebraic group G = Spec S (1.2) is reductive [10].

§2. <u>Quotient and geometric quotient</u>.

In our situation we have a reductive group G = Spec S operating on an affine variety X = Spec R (1.2). Now in [10] Mumford has proved the following.

<u>lemma 2.1</u>. If G is a reductive group acting on a vector space V, then the subspace of invariant elements of V, denoted by V^G, has a unique complement V_G.

<u>theorem 2.2</u>. Let G = Spec S be a reductive group acting on an affine variety X = Spec R by means of an algebra homomorphism $\sigma^* : R \to S \otimes_k R$.

Then the ring of invariants R^G is a k-algebra of finite type. In particular Spec R^G is a variety.

<u>definition 2.3</u>. Let G be a group operating on a variety X. A quotient of X by G is a pair (Y,ϕ) in which Y is a variety and $\phi : X \to Y$ a morphism satisfying:

(1) ϕ is constant on the orbits of the closed points of X and

(2) given a variety Z and a morphism $\psi : X \to Z$, constant on those orbits, there is a unique morphism $\theta : Y \to Z$ such that $\psi = \theta \circ \phi$.

The quotient of X by G is clearly unique up to isomorphism.

<u>definition 2.4</u>. Let G be a group operating on a variety X. A geometric quotient of X by G is a pair (Y,ϕ) consisting of a variety Y and a morphism $\phi : X \to Y$ satisfying:

(1) for each closed point $y \in Y$, $\phi^{-1}(y)$ is an orbit, i.e., a closed invariant subset of X such that G acts transitively on its closed points.

(2) for each invariant open $U \subset X$ there is an open $V \subset Y$ such that $U = \phi^{-1}(V)$.

(3) for each open $V \subset Y$, $\phi^* : \Gamma(V,\mathcal{O}_Y) \to \Gamma(\phi^{-1}(V),\mathcal{O}_X)$ is an isomorphism of $\Gamma(V,\mathcal{O}_Y)$ onto the ring of invariant functions on $\phi^{-1}(V)$.

<u>proposition 2.5</u>. A geometric quotient of a variety by a group is a quotient. In particular it is unique up to isomorphism.

<u>theorem 2.6. (Main-theorem)</u>.

Let G be a reductive affine algebraic group acting algebraically on an affine variety X = Spec R. Then (Y,ϕ), with Y = Spec R^G and ϕ induced by the inclusion $R^G \subset R$, is a quotient of X by G. Moreover:

(1) If x and x' are closed points of x, $\phi(x) = \phi(x')$ iff the intersection of the closure of the orbits of x and x' is not empty.

(2) For each closed point y of Y, $\phi^{-1}(y)$ contains a unique closed orbit.

(3) There is an invariant open subset X_S of X such that a closed point $x \in X$ is in X_S iff the orbit of x is closed and the stabilizer $S(x)$ of x has minimal dimension. Then $Y_S = \phi(X_S)$ is open in Y and $(Y_S, \phi|X_S)$ is a geometric quotient of X_S by G.

<u>From now on the situation we consider is always the same</u>:

$R = k[\underline{X}_1, \ldots, \underline{X}_n]$, X = Spec R (1.1). $n \geq 2$.

$S = k[T_{11}, T_{12}, T_{21}, T_{22}, (T_{11}T_{22} - T_{12}T_{21})^{-1}]$, G = Spec S.

G operates on X by means of the k-algebra homomorphism $\sigma^* : R \to S \otimes_k R$ as defined in 1.2. and R^G is the ring of invariants (1.3).

Y = Spec R^G, $\phi : X \to Y$ is the morphism induced by the inclusion $R^G \subset R$.

If $x \in X$ is a closed point then $x \in X_S$ iff the orbit of x is closed and the stabilizer $S(x)$ of x has minimal dimension. $Y_S = \phi(X_S)$ and $\phi' : X_S \to Y_S$ is the restriction of $\phi : X \to Y$.

Then we know that (Y, ϕ) is a quotient of X by G and (Y_S, ϕ') is a geometric quotient of X_S by G.

Chapter II. The study of X_S.

§1. Reducibility.

definition 1.1. Two elements $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$ of $M(2,k)^n$ are said to be equivalent iff there exists $T \in Gl(2,k)$ such that $B_i = T^{-1}A_iT$ for all $1 \leq i \leq n$. Notation: $(A_1,\dots,A_n) \sim (B_1,\dots,B_n)$. The equivalence class of $(A_1,\dots,A_n)$ is denoted by $cl(A_1,\dots,A_n)$.

definition 1.2. $(A_1,\dots,A_n) \in M(2,k)^n$ is reducible iff $cl(A_1,\dots,A_n)$ contains an element of the form $(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},\dots,\begin{pmatrix} * & * \\ 0 & * \end{pmatrix})$. Equivalent: viewed as linear transformations of k^2, the A_i have a common eigenvector. Irreducible = not reducible.

proposition 1.3. The following properties for $(A,B) \in M(2,k)^2$ are equivalent (cf. [2]):

(1) (A,B) is reducible

(2) If $\{\alpha_1,\alpha_2\}$,$\{\beta_1,\beta_2\}$ and $\{\gamma_1,\gamma_2\}$ are the eigenvalues of A,B and $C = -A-B$ respectively, then $\alpha_i+\beta_j+\gamma_k = 0$ for some $i,j,k \in \{1,2\}$

(3) $Tr(AB)^2 = TrA^2B^2$.

proof:

(1) $\to$ (2): $(A,B) \sim (\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix},\begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix})$ with $\{a_1,a_2\} = \{\alpha_1,\alpha_2\}$ and $\{b_1,b_2\} = \{\beta_1,\beta_2\} \Rightarrow (A,B,C) \sim (\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix},\begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix},\begin{pmatrix} -a_1-b_1 & * \\ 0 & -a_2-b_2 \end{pmatrix}) \Rightarrow$ $\{\gamma_1,\gamma_2\} = \{-a_1-b_1,-a_2,-b_2\}$.

(2) $\to$ (1): By changing, if necessary, the denumbering we may assume $\alpha_1+\beta_1+\gamma_1 = 0$. Because $Tr(A+B+C) = 0$ we also have $\alpha_2+\beta_2+\gamma_2 = 0$.

So $\gamma_i = -\alpha_i - \beta_i$ $(i = 1,2)$. Now assume (A,B) to be irreducible. Let $e_1 \in k^2$ be an eigenvector of A belonging to α_1 and e_2 an eigenvector of B belonging to β_2. Then (e_1, e_2) is a basis of k^2 because if not, A and B would have a common eigenvector. Using this basis we have: $(A,B) \sim \left(\begin{pmatrix}\alpha_1 & x \\ 0 & \alpha_2\end{pmatrix}, \begin{pmatrix}\beta_1 & 0 \\ y & \beta_2\end{pmatrix}\right)$ with $xy \neq 0$ because of the assumed irreducibility of (A,B). So $C = -A-B \sim \begin{pmatrix}-\alpha_1-\beta_1 & -x \\ -y & -\alpha_2-\beta_2\end{pmatrix} = \begin{pmatrix}\gamma_1 & -x \\ -y & \gamma_2\end{pmatrix}$.

But then $\det C = \gamma_1\gamma_2 - xy = \det C - xy$, so $xy = 0$. Contradiction.

(2) $\leftrightarrow$ (3): Define $x_1 = \mathrm{Tr}A$, $x_2 = \mathrm{Tr}B$, $y_1 = \det A$, $y_2 = \det B$ and $z = \mathrm{Tr}AB$.

Then: $\{\alpha_1, \alpha_2\} = \{\tfrac{1}{2}(x_1 + \sqrt{x_1^2 - 4y_1}),\ \tfrac{1}{2}(x_1 - \sqrt{x_1^2-4y_1})\}$

and $\{\beta_1, \beta_2\} = \{\tfrac{1}{2}(x_2 + \sqrt{x_2^2 - 4y_2}),\ \tfrac{1}{2}(x_2 - \sqrt{x_2^2-4y_2})\}$.

$\{\gamma_1, \gamma_2\}$ is the set of zeros of $X^2 + (\mathrm{Tr}A + \mathrm{Tr}B)X + \det(A+B)$. One easily verifies $\det(A+B) = \det A + \det B + \mathrm{Tr}A\mathrm{Tr}B - \mathrm{Tr}AB$, so we have:

$$\{\gamma_1,\gamma_2\} = \{\tfrac{1}{2}(-x_1-x_2 + \sqrt{(x_1+x_2)^2 - 4(y_1+y_2+x_1x_2-z)}),$$
$$\tfrac{1}{2}(-x_1-x_2 - \sqrt{(x_1+x_2)^2 - 4(y_1+y_2+x_1x_2-z)})\}.$$

Now consider the following equivalences:

$$\begin{aligned}\alpha_i+\beta_j+\gamma_k = 0 &\leftrightarrow (\sqrt{x_1^2-4y_1} \pm \sqrt{x_2^2-4y_2})^2 = (x_1+x_2)^2 - 4(y_1+y_2+x_1x_2-z) \\ &\leftrightarrow \pm 2\sqrt{(x_1^2-4y_1)(x_2^2-4y_2)} = 4z - 2x_1x_2 \\ &\leftrightarrow z^2 - 2y_1y_2 = x_1x_2z - x_1^2y_2 - x_2^2y_1 + 2y_1y_2.\end{aligned}$$

We have to show that this is equivalent to $\mathrm{Tr}(AB)^2 = \mathrm{Tr}A^2B^2$. But in virtue of Hamilton-Cayley, i.e., $A^2 = (\mathrm{Tr}A)A - (\det A)I_2$ for every $A \in M(2,k)$, we have: $\mathrm{Tr}(AB)^2 = (\mathrm{Tr}AB)^2 - 2\det A \det B = z^2 - 2y_1y_2$ and also $\mathrm{Tr}A^2B^2 = \mathrm{Tr}[(\mathrm{Tr}A)A - (\det A)I_2][(\mathrm{Tr}B)B - (\det B)I_2] =$

$= x_1x_2z - x_1^2y_2 - x_2^2y_1 + 2y_1y_2$. QED

<u>proposition 1.4</u>. The following properties for $(A,B,C) \in M(2,k)^3$ are equivalent:

(1) (A,B,C) is reducible

(2) (A,B), (A,C), (B,C) are reducible and $\mathrm{Tr}ABC = \mathrm{Tr}ACB$

(3) (A,B), (A,C), (B,C) and $(A,A+B+C)$ are reducible.

<u>proof</u>: (1) $\to$ (2) and (1) $\to$ (3) are evident.

(2) $\to$ (1): Let e_{AB} be a common eigenvector of A and B, and let e_{AC} be a common eigenvector of A and C. If $\dim [\![e_{AB},e_{AC}]\!] = 1$ one has $e_{AC} = \lambda e_{AB}$ for some $\lambda \in k^*$, but then e_{AB} is a common eigenvector of A, B and C so (A,B,C) is reducible. If $\dim [\![e_{AB},e_{AC}]\!] = 2$, then using (e_{AB},e_{AC}) as a basis of k^2 we get: $(A,B,C) \sim ((\begin{smallmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{smallmatrix}),(\begin{smallmatrix}\beta_1 & x\\ 0 & \beta_2\end{smallmatrix}),(\begin{smallmatrix}\gamma_1 & 0\\ y & \gamma_2\end{smallmatrix}))$.
Now, $0 = \mathrm{Tr}ABC - \mathrm{Tr}ACB = (\alpha_1-\alpha_2)xy$ and so there are two cases.

Case 1: $xy = 0 \to x = 0$ or $y = 0 \to (A,B,C)$ is reducible

Case 2: $\alpha_1 = \alpha_2$. Then $(A,B,C) = (\alpha_1 I_2,B,C)$. Because of the reducibility of (B,C) it then follows that (A,B,C) is reducible too.

(3) $\to$ (1): Let $D = A+B+C$. Define e_{AB} to be a common eigenvector of A and B. Similarly one defines e_{AC}, e_{BC} and e_{AD}.
If $\dim [\![e_{AB},e_{AC}]\!] = 1$ we are ready. So suppose $\dim [\![e_{AB},e_{AC}]\!] = 2$.
If $e_{AD} \in [\![e_{AB}]\!]$ (or $e_{AD} \in [\![e_{AC}]\!]$) then e_{AB} (or e_{AC}) is a common eigenvector of A,B and A+B+C (or A,C and A+B+C respectively), whence also a common eigenvector of A,B and C. In these cases we are ready.
What remains is the case $e_{AD} = \lambda e_{AB}+\mu e_{AC}$ with $\lambda,\mu \neq 0$. Say e_{AB}, e_{AC} and e_{AD} belong to the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ of A. Then we have:
$Ae_{AD} = \lambda_3 e_{AD} = \lambda_3\lambda e_{AB}+\lambda_3\mu e_{AC}$ and $Ae_{AD} = A(\lambda e_{AB}+\mu e_{AC}) = \lambda_1\lambda e_{AB}+\lambda_2\mu e_{AC}$
$\to \lambda_3\lambda = \lambda_1\lambda,\ \lambda_3\mu = \lambda_2\mu \to \lambda_1 = \lambda_2 = \lambda_3 \to A = \lambda_1 I_2 \to (A,B,C)$ is reducible because of the reducibility of (B,C).

QED.

<u>remark</u>: The conditions in (2) of prop.1.4. are necessary and sufficient:

(i) Let $(A,B,C) = ((\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}),(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}),(\begin{smallmatrix}0 & 0\\ 1 & 1\end{smallmatrix}))$ then the pairs (A,B), (A,C)

and (B,C) are reducible but (A,B,C) is not.

(ii) Let $(A,B,C) = (\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix})$ then (A,B) and (A,C) are reducible, TrABC = TrACB, but (A,B,C) is irreducible.

<u>proposition 1.5</u>. If $n \geq 3$ and $(A_1,\ldots,A_{n+1}) \in M(2,k)^{n+1}$ then $(A_1,\ldots,A_{n+1})$ is reducible iff each n-tuple $(A_1,\ldots,\hat{A}_i,\ldots,A_n)$ is reducible $(1 \leq i \leq n)$.

<u>proof</u>: Suppose each n-tuple to be reducible and let $e_i \in k^2$ be a common eigenvector of $A_1,\ldots,\hat{A}_i,\ldots,A_n$ $(1 \leq i \leq n)$. We have to prove that $(A_1,\ldots,A_{n+1})$ is reducible. Assume that $(A_1,\ldots,A_{n+1})$ is irreducible, then $\dim [\![e_1,\ldots,e_{n+1}]\!] \neq 1$, otherwise we would have a common eigenvector of $A_1,\ldots,A_{n+1}$. So $\dim [\![e_1,\ldots,e_{n+1}]\!] = 2$. Say (e_{i_1},e_{i_2}) with $1 \leq i_1 < i_2 \leq n+1$ is a basis of k^2. Using this basis we get:

$$(A_{i_1},A_{i_2},A_1,\ldots,\hat{A}_{i_1},\ldots,\hat{A}_{i_2},\ldots,A_{n+1}) \sim (\begin{pmatrix}\alpha_1 & 0\\x & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & y\\0 & \beta_2\end{pmatrix},\begin{pmatrix}\lambda_1 & 0\\0 & \mu_1\end{pmatrix},\ldots,\begin{pmatrix}\lambda_{n-1} & 0\\0 & \mu_{n-1}\end{pmatrix}).$$

Becfuse each n-tuple is reducible and $n \geq 3$ we certainly have the reducibility of the triple $(P,Q,R) = (\begin{pmatrix}\alpha_1 & 0\\x & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & y\\0 & \beta_2\end{pmatrix},\begin{pmatrix}\lambda_1 & 0\\0 & \mu_1\end{pmatrix})$.

Using prop.1.4. we obtain $0 = \mathrm{Tr}PQR-\mathrm{Tr}PRQ = (\mu_1-\lambda_1)xy$. Now there are two possibilities:

(1) $xy = 0 \rightarrow x = 0$ or $y = 0 \rightarrow (A_1,\ldots,A_{n+1})$ is reducible. Contradiction.

(2) $\lambda_1 = \mu_1$. Then $A_p = \lambda_1 I_2$ for some $p \in \{1,\ldots,n+1\}$ which implies the reducibility of $(A_1,\ldots,A_{n+1})$ in view of the reducibility of $(A_1,\ldots,\hat{A}_p,\ldots,A_{n+1})$.

QED.

<u>corollary 1.6</u>. If $n \geq 4$ and $(A_1,\ldots,A_n) \in M(2,k)^n$ then $(A_1,\ldots,A_n)$ is reducible iff each triple $(A_{i_1},A_{i_2},A_{i_3})$ taken from $\{A_1,\ldots,A_n\}$ is reducible.

<u>proof</u>: evident.

§2. The separation property.

definition 2.1. $\mathcal{B}_n = \{cl(A_1,\dots,A_n) \mid (A_1,\dots,A_n) \in M(2,k)^n \text{ irreducible}\}$. Define the map $\bar{i} : \mathcal{B}_n \to k^m$ by $\bar{i}(cl(A_1,\dots,A_n)) = i(A_1,\dots,A_n)$. (cf. I.1.9).

In this § we prove that the map $\bar{i}$ is injective, i.e., two irreducible elements of $M(2,k)^n$ have the same invariants iff they are equivalent to each other.

lemma 2.2. (cf. Dekkers [2]). Let (A,B) be an irreducible element of $M(2,k)^2$. Let $\{\alpha_1,\alpha_2\}$ and $\{\beta_1,\beta_2\}$ be the eigenvalues of A and B respectively, then $(A,B) \sim (\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix})$ where $\rho = TrAB-\alpha_1\beta_1-\alpha_2\beta_2$. Moreover: $\rho(TrAB-\alpha_1\beta_2-\alpha_2\beta_1) = Tr(AB)^2-TrA^2B^2 \neq 0$.

proof: Let e_1 be an eigenvector of A belonging to α_1 and e_2 an eigenvector of B belonging to β_2. Because (A,B) is irreducible, (e_1,e_2) is a basis of k^2. Using this basis one gets: $(A,B) \sim (\begin{pmatrix}\alpha_1 & x\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ y & \beta_2\end{pmatrix})$ with $xy \neq 0$. So choosing (xe_1,e_2) as a basis of k^2 we have:

$$(A,B) \sim (\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix}) \text{ for some } \rho \in k^*$$

$$TrAB = Tr\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix} = \alpha_1\beta_1+\rho+\alpha_2\beta_2 \to \rho = TrAB-\alpha_1\beta_1-\alpha_2\beta_2$$

$$\begin{aligned}\rho(TrAB-\alpha_1\beta_2-\alpha_2\beta_1) &= (TrAB)^2-(\alpha_1+\alpha_2)(\beta_1+\beta_2)TrAB+(\alpha_1+\alpha_2)^2\beta_1\beta_2+\\ &\quad +(\beta_1+\beta_2)^2\alpha_1\alpha_2-4\alpha_1\alpha_2\beta_1\beta_2 =\\ &= (TrAB)^2-TrATrBTrAB+(TrA)^2\det B+(TrB)^2\det A-4\det A\det B =\\ &= Tr(AB)^2-TrA^2B^2 \text{ (cf. the proof of 1.3).}\end{aligned}$$

proposition 2.3. $\bar{i} : \mathcal{B}_2 \to k^5$ is injective.

proof: Let (A,B) and (C,D) be irreducible elements of $M(2,k)^2$ and assume $i(A,B) = i(C,D) = (a_1,a_2,a_3,a_4,a_5)$. Then the eigenvalues of A and C are the same because they are the solutions of $X^2-a_1X-\frac{1}{4}(a_3-a_1^2)=0$

(remember $a_1 = \mathrm{Tr}A$, $a_3 = 2\mathrm{Tr}A^2-(\mathrm{Tr}A)^2$); let us call them α_1 and α_2.
Analogously: B and D have the same eigenvalues β_1 and β_2.
In view of the above lemma we have: $(A,B) \sim (\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix})$ and $(C,D) \sim (\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho' & \beta_2\end{pmatrix})$. But $\rho = \mathrm{Tr}AB-\alpha_1\beta_1-\alpha_2\beta_2 = \mathrm{Tr}CD-\alpha_1\beta_1-\alpha_2\beta_2 = \rho'$, so $\mathrm{cl}(A,B) = \mathrm{cl}(C,D)$.

<u>lemma 2.4</u>. Let (A,B) be an irreducible element of $M(2,k)^2$; let $i : M(2,k)^3 \to k^{10}$ be the invariant-map (I.1.9). If $C,D \in M(2,k)$ such that $i(A,B,C) = i(A,B,D)$ then $C = D$.

<u>proof</u>: Using lemma 2.2 we can assume $(A,B,C) = (\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix},\begin{pmatrix}a & b\\ c & d\end{pmatrix})$.
We are ready if we prove that the coefficients a,b,c and d of C are determined by $i(A,B,C) \in k^{10}$.

Say $i(A,B,C) = (x_1,x_2,x_3,y_1,y_2,y_3,z_{12},z_{13},z_{23},w)$.

Then:
$$\begin{aligned} x_3 &= \mathrm{Tr}C = a+d\\ z_{13} &= 2\mathrm{Tr}AC-\mathrm{Tr}A\mathrm{Tr}C = (\alpha_1-\alpha_2)(a-d)+2c\\ z_{23} &= 2\mathrm{Tr}BC-\mathrm{Tr}B\mathrm{Tr}C = (\beta_1-\beta_2)(a-d)+2\rho b\\ w &= \mathrm{Tr}ABC-\mathrm{Tr}ACB = -(\alpha_1-\alpha_2)\rho b-(\beta_1-\beta_2)c+\rho(a-d). \end{aligned}$$

Or, equivalently:
$$\begin{pmatrix} x_3\\ z_{13}\\ z_{23}\\ -w\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \alpha_1-\alpha_2 & 2 & 0\\ 0 & \beta_1-\beta_2 & 0 & 2\\ 0 & -\rho & \beta_1-\beta_2 & \alpha_1-\alpha_2 \end{pmatrix}\begin{pmatrix} a+d\\ a-d\\ c\\ \rho b\end{pmatrix}$$

The matrix up here is invertible, because its determinant is equal to $-4(\rho+(\alpha_1-\alpha_2)(\beta_1-\beta_2)) = -4(\mathrm{Tr}AB-\alpha_1\beta_2-\alpha_2\beta_1) = -4\rho^{-1}[\mathrm{Tr}(AB)^2-\mathrm{Tr}A^2B^2] \neq 0$ (cf. 2.2). So a+d, a-d, c and ρb are determined by $i(A,B,C)$ and therefore also a,b,c and d.

<u>corollary 2.5</u>. If (A,B) is irreducible and $i(A,B,C_1,\ldots,C_p) = i(A,B,D_1,\ldots,D_p)$ then $C_j = D_j$ for all $1 \leq j \leq p$.

<u>proof</u>: This is an immediate consequence of lemma 2.4 because it is

clear that for all $1 \leq j \leq p$ we have $i(A,B,C_j) = i(A,B,D_j)$.

<u>proposition 2.6</u>. For all $n \geq 2$ $\bar{i} : \mathcal{B}_n \to k^m$ is injective.

<u>proof</u>: (i) $n = 2$: prop.2.3.

(ii) If (A,B) is irreducible then (a) $i(A,B,C_1,\ldots,C_p) =$
$= i(A',B',C_1',\ldots,C_p')$ and (b) $(A,B,C_1,\ldots,C_p) \sim (A',B',C_1',\ldots,C_p')$ are equivalent. This is seen as follows:

(b) $\to$ (a) is evident.

(a) $\to$ (b): $i(A,B) = i(A',B')$ so, with prop.2.3, we have $(A,B) \sim (A',B')$. (Notice that (A',B') also has to be irreducible because reducibility is equivalent with the vanishing of some expression in the invariants). Say $(A',B') = (T^{-1}AT, T^{-1}BT)$. Define $D_j = TC_j'T^{-1}$ $(1 \leq j \leq p)$, then
$(A',B',C_1',\ldots,C_p') = (T^{-1}AT, T^{-1}BT, T^{-1}D_1T,\ldots,T^{-1}D_pT) \sim (A,B,D_1,\ldots,D_p)$.
Now $i(A,B,D_1,\ldots,D_p) = i(A',B',C_1',\ldots,C_p') = i(A,B,C_1,\ldots,C_p)$, so $D_j = C_j$ (cor.2.5) which, in turn, implies $(A',B',C_1',\ldots,C_p') \sim (A,B,C_1,\ldots,C_p)$.

(iii) $n \geq 3$. Let $(A_1,\ldots,A_n)$ and $(B_1,\ldots,B_n)$ be irreducible and $i(A_1,\ldots,A_n) = i(B_1,\ldots,B_n)$. From cor.1.6 we know that there exist $i_1,i_2,i_3 \in \{1,\ldots,n\}$ such that $(A_{i_1},A_{i_2},A_{i_3})$ is irreducible. Without restriction we may assume (A_1,A_2,A_3) to be irreducible. With prop.1.4. in mind we distinguish four possibilities:

(1) (A_1,A_2) is irreducible. Then complete the proof using (ii).

(2) (A_1,A_3) is irreducible. Notice $i(A_1,A_3,A_2,A_4,\ldots,A_n) =$
$= i(B_1,B_3,B_2,B_4,\ldots,B_n)$ and use (ii) again.

(3) (A_2,A_3) is irreducible. Then $i(A_2,A_3,A_1,A_4,\ldots,A_n) =$
$= i(B_2,B_3,B_1,B_4,\ldots,B_n)$ so, using (ii), again we are ready.

(4) $(A_1,A_1+A_2+A_3)$ is irreducible. Then $i(A_1,A_1+A_2+A_3,A_3,\ldots,A_n) =$
$= i(B_1,B_1+B_2+B_3,B_3,\ldots,B_n)$ implies

$(A_1, A_1+A_2+A_3, A_3, \dots, A_n) \sim (B_1, B_1+B_2+B_3, B_3, \dots, B_n)$, from which we obtain $(A_1, \dots, A_n) \sim B_1, \dots, B_n)$.

§3. The study of X_S.

$X_S \subset X = \text{Spec } R$. A closed point $x \in X$ is in X_S iff the orbit of x is closed and its stabilizer $S(x)$ has minimal dimension.

proposition 3.1. Let $x \in X$ be a closed point corresponding to $(A_1, \dots, A_n) \in M(2,k)^n$. Then the orbit of x is closed if and only if either $(A_1, \dots, A_n)$ is irreducible or $(A_1, \dots, A_n)$ is diagonalizable (i.e. $cl(A_1, \dots, A_n)$ contains an element $(D_1, \dots, D_n)$ where the D_i are diagonal matrices).

proof:

(i) Assume the orbit of x to be closed and $(A_1, \dots, A_n)$ neither irreducible nor diagonalizable. In both cases $(A_1, \dots, A_n)$ is reducible, so there are $a_i, b_i, c_i \in k$ $(1 \le i \le n)$ satisfying $(A_1, \dots, A_n) \sim (\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & b_n \\ 0 & c_n \end{pmatrix})$. Conjugating with $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ one obtains $(A_1, \dots, A_n) \sim (\begin{pmatrix} a_1 & tb_1 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & tb_n \\ 0 & c_n \end{pmatrix})$ for all $t \in k^*$. But then also $(\begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & 0 \\ 0 & c_n \end{pmatrix}) \in cl(A_1, \dots, A_n)$ because the orbit of x is closed.

(ii) First consider the irreducible case.

$\alpha : \Sigma \to R$ (cf. I.1.7) induces the morphism $\psi : X = \text{Spec } R \to \mathbb{A}^m$.

I claim that the orbit of x is equal to $\psi^{-1}\psi(x)$ (and therefore closed). This is seen as follows: let x', a closed point of X, be in the orbit of x. Say x' corresponds to $(B_1, \dots, B_n)$.

$$
\begin{aligned}
x' \text{ is in the orbit of } x &\leftrightarrow (A_1, \dots, A_n) \sim (B_1, \dots, B_n) \\
&\leftrightarrow i(A_1, \dots, A_n) = i(B_1, \dots, B_n) \text{ (using prop.2.6)} \\
&\leftrightarrow \psi(x) = \psi(x') \\
&\leftrightarrow x' \in \psi^{-1}\psi(x)
\end{aligned}
$$

(iii) Now suppose that $(A_1,\dots,A_n)$ is diagonalizable.

Say $(A_1,\dots,A_n) \sim (\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix})$ and $\begin{pmatrix}\alpha_i & 0\\ 0 & \beta_i\end{pmatrix} = UA_iU^{-1}$ for all $1 \le i \le n$.

Distinguish two cases:

(a) If $\alpha_i = \beta_i$ for all $1 \le i \le n$ then $(A_1,\dots,A_n) = (\alpha_1 I_2,\dots,\alpha_n I_2)$ so $\{x\}$ is the orbit of x and therefore this orbit is closed.

(b) If $\alpha_i \neq \beta_i$ for some i we take for convenience $i = 1$. Let $\theta : X \to \mathbb{A}^4$ be the morphism induced by the inclusion $k[\underline{X}_1] \subset R = k[\underline{X}_1,\dots,\underline{X}_n]$. We claim that the orbit of x equals $\theta^{-1}\theta(x)$. We only have to prove that the projection on the first factor $p_1 : M(2,k)^n \to M(2,k)$ induces a bijection between the sets $cl(A_1,\dots,A_n)$ and $cl\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}$.

Well, it is clear that p_1 maps $cl(A_1,\dots,A_n)$ onto $cl\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}$, because $A_1 \sim' \begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}$. Now suppose $p_1(S^{-1}A_1 S,\dots,S^{-1}A_nS) =$
$= p_1(T^{-1}A_1T,\dots,T^{-1}A_nT)$, i.e., $S^{-1}A_1S = T^{-1}A_1T$. Then we have $S^{-1}U^{-1}\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}US = T^{-1}U^{-1}\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}UT$, or $UTS^{-1}U^{-1}\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}UST^{-1}U^{-1} = \begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}$.
This implies $UTS^{-1}U^{-1} = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ with $\lambda\mu \neq 0$. But then $UTS^{-1}U^{-1}\begin{pmatrix}\alpha_i & 0\\ 0 & \beta_i\end{pmatrix}UST^{-1}U^{-1} = \begin{pmatrix}\alpha_i & 0\\ 0 & \beta_i\end{pmatrix}$ for all $1 \le i \le n$, from which it follows that $S^{-1}A_iS = T^{-1}A_iT$ for all $1 \le i \le n$.

QED

<u>proposition 3.2</u>. Let $x \in X$ be a closed point corresponding to $(A_1,\dots,A_n)$. Then $x \in X_S$ if and only if $(A_1,\dots,A_n)$ is irreducible.

<u>proof</u>: Taking into account prop.3.1 it is sufficient to prove:

If $A = (A_1,\dots,A_n) \in M(2,k)^n$ and $S(A)$ denotes the stabilizer of A, then

(a) $S(A) = \{\lambda I_2 \mid \lambda \in k^*\}$ if A is irreducible, so $\dim S(x) = 1$

(b) $\dim S(x) \geq 2$ if A is diagonalizable.

(a) If A is irreducible and $(T^{-1}A_1T,\dots,T^{-1}A_nT) = (A_1,\dots,A_n)$ we regard

$(A_1,\dots,A_n)$ as an irreducible family of endomorphisms of k^2. T commutes with each A_i. Let λ be an eigenvalue of T and $B = T-\lambda I$ then $B : k^2 \to k^2$ is not injective. Therefore dim (ker B) ≥ 1. Now $A_iB = BA_i$ implies $A_i(\ker B) \subset \ker B$. In consequence of the irreducibility of $(A_1,\dots,A_n)$ we then conclude that ker B equals k^2 or equivalently $T = \lambda I$ (cf. Schur's lemma).

(b) If A is diagonalizable there exist $U \in Gl(2,k)$ and $\alpha_i,\beta_i \in k$ such that $UA_iU^{-1} = \begin{pmatrix}\alpha_i & 0\\ 0 & \beta_i\end{pmatrix}$ for all $1 \leq i \leq n$.

Now: $T \in S(A) \leftrightarrow T^{-1}A_iT = A_i$ for all $i \leftrightarrow UTU^{-1} \in S(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix})$.

If $\alpha_i = \beta_i$ for all $1 \leq i \leq n$ then $S(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix}) = Gl(2,k)$ and if $\alpha_i \neq \beta_i$ for some i we have $S(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix}) = \{\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} \mid \lambda\mu \neq 0\}$.

So in both cases we have $\dim S(x) \geq 2$.

Chapter III. The cases n = 2 and n = 3.

Dekkers has studied the case n = 2 [2]. In §1 we repeat his results, using our notation. In §2 we treat, without detailed proof, the case n = 3 in order to illustrate the origin of our earlier conjecture and present theorem that Y_S is the smooth part of Y in case $n \geq 3$.

§1. The case n = 2.

We have the following situation:

X = Spec R, where $R = k[\underline{X}_1,\underline{X}_2]$. X_S corresponds to the irreducible elements of $M(2,k)^2$. Bearing in mind II.1.3, X_S is the open part of X defined by the inequality $\mathrm{Tr}(\underline{X}_1\underline{X}_2)^2 - \mathrm{Tr}\underline{X}_1^2\underline{X}_2^2 \neq 0$.

$\alpha : \Sigma = k[X_1,X_2,Y_1,Y_2,Z_{12}] \to R$; $\alpha(\Sigma) = R^G$; Y = Spec R^G.

Furthermore the diagram below is commutative.

$$\begin{array}{ccl} X_S & \hookrightarrow & X = \mathrm{Spec}\ R \\ {\scriptstyle \phi'}\downarrow & & \quad\ \downarrow{\scriptstyle \phi} \\ Y_S & \hookrightarrow & Y = \mathrm{Spec}\ R^G \simeq \mathrm{Spec}\ \Sigma/\ker\alpha \end{array}$$

<u>lemma 1.1</u>. $i : M(2,k)^2 \to k^5$ is surjective (cf. I.1.9).

<u>proof</u>: Let $a = (x_1,x_2,y_1,y_2,z) \in k^5$. Choose $\alpha_1,\alpha_2,\beta_1,\beta_2$ and ρ in k as follows: α_1 and α_2 are the solutions of $X^2-x_1X-\frac{1}{4}(y_1-x_1^2) = 0$, β_1 and β_2 those of $X^2-x_2X-\frac{1}{4}(y_2-x_2^2) = 0$ and $\rho = \frac{1}{2}z-\frac{1}{2}(\alpha_1-\alpha_2)(\beta_1-\beta_2)$.

Define $(A,B) = (\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix})$, then one has:

$\mathrm{Tr}A = \alpha_1+\alpha_2 = x_1$

$\mathrm{Tr}B = \beta_1+\beta_2 = x_2$

$2\mathrm{Tr}A^2-(\mathrm{Tr}A)^2 = 2(\alpha_1^2+\alpha_2^2)-x_1^2 = 2(\alpha_1+\alpha_2)^2-4\alpha_1\alpha_2-x_1^2 = y_1$

$2\mathrm{Tr}B^2-(\mathrm{Tr}B)^2 = y_2$

$2\mathrm{Tr}AB - \mathrm{Tr}A\mathrm{Tr}B = 2(\alpha_1\beta_1+\rho+\alpha_2\beta_2)-(\alpha_1+\alpha_2)(\beta_1+\beta_2) = z.$

And therefore $i(A,B) = a$.

<u>proposition 1.2</u>. $\ker \alpha = \{0\}$, so $\Sigma \simeq R^G$ and $Y \simeq \mathbb{A}^5$.

<u>proof</u>:

$$\begin{aligned} g \in \ker\alpha &\leftrightarrow \tilde{A}(\alpha(g)) = 0 \text{ for all } A \in M(2,k)^2 \\ &\leftrightarrow \widetilde{i(A)}(g) = 0 \text{ for all } A \in M(2,k)^2 \quad \text{(cf. I.1.10)} \\ &\leftrightarrow \tilde{a}(g) = 0 \text{ for all } a \in k^5 \quad \text{(lemma 1.1)} \\ &\leftrightarrow g = 0. \end{aligned}$$

<u>proposition 1.3</u>. (a) $\alpha(Z_{12}^2-Y_1Y_2) = 4[\mathrm{Tr}(\underline{X}_1\underline{X}_2)^2-\mathrm{Tr}\underline{X}_1^2\underline{X}_2^2]$

(b) $Y-Y_S \simeq V(Z_{12}^2-Y_1Y_2) \subset \mathbb{A}^5 = \mathrm{Spec}\,\Sigma$.

<u>proof</u>: (a) implies (b) in view of II.1.3, II.3.2 and the definition of Y_S.

In $M(2,R)$ we evidently have: $\underline{X}_i^2 = (\mathrm{Tr}\underline{X}_i)\underline{X}_i - (\det \underline{X}_i)I_2$ $(i = 1,2)$ and also $(\underline{X}_1\underline{X}_2)^2 = (\mathrm{Tr}\underline{X}_1\underline{X}_2)\underline{X}_1\underline{X}_2 - (\det \underline{X}_1 \det \underline{X}_2)I_2$ (Hamilton-Cayley).

From these identities it follows:

$2\det \underline{X}_i = (\mathrm{Tr}\underline{X}_i)^2-\mathrm{Tr}\underline{X}_i^2$ $(i = 1,2)$

$\mathrm{Tr}(\underline{X}_1\underline{X}_2)^2 = (\mathrm{Tr}\underline{X}_1\underline{X}_2)^2 - 2\det \underline{X}_1 \det \underline{X}_2$

$\mathrm{Tr}\underline{X}_1^2\underline{X}_2^2 = \mathrm{Tr}\underline{X}_1\,\mathrm{Tr}\underline{X}_2\,\mathrm{Tr}\underline{X}_1\underline{X}_2 - (\mathrm{Tr}\underline{X}_1)^2\det \underline{X}_2-(\mathrm{Tr}\underline{X}_2)^2\det \underline{X}_1 + 2\det \underline{X}_1 \det \underline{X}_2.$

By definition of $\alpha : \Sigma \to R$ we have:

$$\begin{aligned} \alpha(Z_{12}^2-Y_1Y_2) &= (2\mathrm{Tr}\underline{X}_1\underline{X}_2-\mathrm{Tr}\underline{X}_1\mathrm{Tr}\underline{X}_2)^2-[2\mathrm{Tr}\underline{X}_1^2-(\mathrm{Tr}\underline{X}_1)^2][2\mathrm{Tr}\underline{X}_2^2-(\mathrm{Tr}\underline{X}_2)^2] = \\ &= (2\mathrm{Tr}\underline{X}_1\underline{X}_2-\mathrm{Tr}\underline{X}_1\mathrm{Tr}\underline{X}_2)^2-[(\mathrm{Tr}\underline{X}_1)^2-4\det \underline{X}_1][(\mathrm{Tr}\underline{X}_2)^2-4\det \underline{X}_2] = \\ &= 4(\mathrm{Tr}\underline{X}_1\underline{X}_2)^2-4\mathrm{Tr}\underline{X}_1\mathrm{Tr}\underline{X}_2\mathrm{Tr}\underline{X}_1\underline{X}_2+4(\mathrm{Tr}\underline{X}_1)^2\det \underline{X}_2 + \\ &\qquad + 4(\mathrm{Tr}\underline{X}_2)^2\det \underline{X}_1 - 16\det \underline{X}_1 \det \underline{X}_2 = \\ &= 4[\mathrm{Tr}(\underline{X}_1\underline{X}_2)^2-\mathrm{Tr}\underline{X}_1^2\underline{X}_2^2]. \end{aligned}$$

§2. The case n = 3.

$\alpha : \Sigma = k[X_1,X_2,X_3,Y_1,Y_2,Y_3,Z_{12},Z_{13},Z_{23},W_{123}] \to R = k[\underline{X}_1,\underline{X}_2,\underline{X}_3]$.

Looking back at the proof of lemma II.2.4, we assumed (A,B) to be irreducible and then $(A,B,C) \sim \left(\begin{pmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{pmatrix},\begin{pmatrix}\beta_1 & 0\\ \rho & \beta_2\end{pmatrix},\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)$.

In proving that a,b,c and d were determined by the invariants $(x_1,x_2,x_3,y_1,y_2,y_3,z_{12},z_{13},z_{23},w)$ of (A,B,C) we did not use the relation $y_3 = 2\mathrm{Tr}C^2-(\mathrm{Tr}C)^2$. If one computes this identity, inserting the expression of a,b,c and d in the invariants, one discovers that i(A,B,C) is a zero of $f \in \Sigma$ defined by:

$f = 4W_{123}^2-Y_1Z_{23}^2-Y_2Z_{13}^2-Y_3Z_{12}^2+2Z_{12}Z_{13}Z_{23}+Y_1Y_2Y_3$. (This relation can also be found in [5], p.366).

Because the irreducibility of (A,B) is equivalent to $i(A,B,C) \notin V(Z_{12}^2-Y_1Y_2)$, it follows from the principle of the irrelevance of algebraic inequalities that $i(A,B,C) \in V(f)$ for <u>all</u> $(A,B,C) \in M(2,k)^3$. Therefore $i : M(2,k)^3 \to V(f) \subset k^{10}$. Just as in §1 above one can prove that $i : M(2,k)^3 \to V(f)$ is surjective.

<u>proposition 2.1</u>. ker α is the ideal of Σ generated by f. In particular $Y \simeq \mathrm{Spec}\ ^{\Sigma}/(f)$.

<u>proof</u>: One easily checks that f is an irreducible element of Σ. Consequently (f) is a prime ideal and therefore $(f) = \sqrt{(f)}$.

Now: $g \in \ker\alpha \Leftrightarrow \alpha(g) = 0 \Leftrightarrow \tilde{A}(\alpha(g)) = 0$ for all $A \in M(2,k)^3 \Leftrightarrow \widetilde{i(A)}(g)=0$ for all A $\Leftrightarrow \tilde{x}(g) = 0$ for all $x \in V(f) \Leftrightarrow g \in \sqrt{(f)} = (f)$ (Nullstellensatz).

<u>definition 2.2.</u> $V(\Delta f)$ denotes the subset of k^{10} consisting of the common zeros of all the partial derivatives of f.

<u>theorem 2.3</u>. Let $(A,B,C) \in M(2,k)^3$, then (A,B,C) is reducible iff $i(A,B,C) \in V(\Delta f)$.

proof: If $i < j$ then $\alpha(Z_{ij}^2 - Y_iY_j) = 4\mathrm{Tr}(\underline{X}_i\underline{X}_j)^2 - 4\mathrm{Tr}\underline{X}_i^2\underline{X}_j^2$ (cf. 1.3). In view of II.1.4.(2) it then follows:

(*): (A,B,C) reducible $\Leftrightarrow i(A,B,C) \in V(W_{123}, Z_{12}^2-Y_1Y_2, Z_{13}^2-Y_1Y_3, Z_{23}^2-Y_2Y_3) \cap V(f)$.

Now by definition of $V(\Delta f)$ we have:

$$V(\Delta f) = V(W_{123}, Z_{12}^2-Y_1Y_2, Z_{13}^2-Y_1Y_3, Z_{23}^2-Y_2Y_3, Z_{12}Z_{13}-Y_1Z_{23}, Z_{12}Z_{23}-Y_2Z_{13}, Z_{13}Z_{23}-Y_3Z_{12}).$$

But consider the following identities:

(1): $(Z_{12}Z_{13}-Y_1Z_{23})^2 = -Y_1f+4Y_1W_{123}^2+(Z_{12}^2-Y_1Y_2)(Z_{13}^2-Y_1Y_3)$

(2): $(Z_{12}Z_{23}-Y_2Z_{13})^2 = -Y_2f+4Y_2W_{123}^2+(Z_{12}^2-Y_1Y_2)(Z_{23}^2-Y_2Y_3)$

(3): $(Z_{13}Z_{23}-Y_3Z_{12})^2 = -Y_3f+4Y_3W_{123}^2+(Z_{13}^2-Y_1Y_3)(Z_{23}^2-Y_2Y_3)$

(4): $f = 4W_{123}^2-Y_3(Z_{12}^2-Y_1Y_2)-Y_2(Z_{13}^2-Y_1Y_3)+Y_1(Z_{23}^2-Y_2Y_3)+2Z_{23}(Z_{12}Z_{13}-Y_1Z_{23})$.

Then we finally have:

$$\begin{aligned} V(\Delta f) &= V(W_{123}, Z_{12}^2-Y_1Y_2, Z_{13}^2-Y_1Y_3, Z_{23}^2-Y_2Y_3, Z_{12}Z_{13}-Y_1Y_{23}, Z_{12}Z_{23}-Y_2Z_{13}, \\ &\qquad Z_{13}Z_{23}-Y_3Z_{12}, f) \\ &= V(W_{123}, Z_{12}^2-Y_1Y_2, Z_{13}^2-Y_1Y_3, Z_{23}^2-Y_2Y_3) \cap V(f). \end{aligned}$$

Combining this with (*) above we obtain the desired result.

remark: From (4) above it follows that $V(\Delta f) \subset V(f)$.

corollary 2.4. Y_S is the smooth part of Y $(n = 3)$.

proof: $Y_S \simeq V(f)-V(\Delta f)$ is the smooth part of $Y \simeq V(f)$.

Chapter IV. Generators of ker α. The smooth part of Y.

§1. Some elements of ker α; the ideal J.

Looking back at §2 of chapter II one sees that, proving the injectivity of $\overline{i}$, we in fact reduced it to the injectivity of $\overline{i} : B_2 \to k^5$. In other words: in proving properties of irreducible elements $(A_1,\ldots,A_n) \in M(2,k)^n$ we often want to reduce it to the case that (A_1,A_2) is irreducible.

This process of reduction consists of two steps. The first step is the reduction to the case that (A_1,A_2,A_3) is irreducible, which can be achieved by a permutation of the indices in $(A_1,\ldots,A_n)$. This step is formalized in definition 1.2. Now the irreducibility of (A_1,A_2,A_3) is equivalent to that of one of the pairs $(A_1,A_2),(A_1,A_3),(A_2,A_3)$ or $(A_1,A_1+A_2+A_3)$. To handle the case of $(A_1,A_1+A_2+A_3)$ we introduce the map $\overline{d} : M(2,k)^n \to M(2,k)^n$ of definition 1.5.4 and also the maps which correspond to $\overline{d}$ on the formal level.

<u>definition 1.1</u>. Σ' is the polynomial ring over k in the $2n+2\binom{n}{2}+6\binom{n}{3}$ indeterminates X_i $(1 \leq i \leq n)$, Y_i $(1 \leq i \leq n)$, Z_{ij} $(1 \leq i \leq n,$ $1 \leq j \leq n,\ i \neq j)$ and W_{ijk} (i,j and k are three different integers between 1 and n). $j : \Sigma \to \Sigma'$ is the inclusion. $\pi : \Sigma' \to \Sigma$ is the surjective k-algebra homomorphism such that:

$$\pi(X_i) = X_i$$
$$\pi(Y_i) = Y_i$$
$$\pi(Z_{ij}) = Z_{ij} \text{ if } i < j$$
$$\pi(Z_{ij}) = Z_{ji} \text{ if } i > j$$
$$\pi(W_{ijk}) = \pi(W_{jki}) = \pi(W_{kij})$$
$$\pi(W_{ijk}) = W_{ijk} \text{ if } i < j < k$$
$$\pi(W_{ijk}) = -W_{ikj} \text{ if } i < k < j$$

remark: (1) $\pi j = \mathrm{id}_\Sigma$

(2) For all different $i,j,k \in \{1,\dots,n\}$ we have:

$(\alpha\pi)(X_i) = \mathrm{Tr}\underline{X}_i$

$(\alpha\pi)(Y_i) = 2\mathrm{Tr}\underline{X}_i^2 - (\mathrm{Tr}\underline{X}_i)^2$

$(\alpha\pi)(Z_{ij}) = 2\mathrm{Tr}\underline{X}_i\underline{X}_j - \mathrm{Tr}\underline{X}_i\mathrm{Tr}\underline{X}_j$

$(\alpha\pi)(W_{ijk}) = \mathrm{Tr}\underline{X}_i\underline{X}_j\underline{X}_k - \mathrm{Tr}\underline{X}_i\underline{X}_k\underline{X}_j.$

definition 1.2. The actions of S_n (the group of permutations of $\{1,\dots,n\}$).

Let $\sigma \in S_n$. Then define:

1.2.1. $\overline{\nu}_\sigma : M(2,k)^n \to M(2,k)^n$ is the map such that

$\overline{\nu}_\sigma(A_1,\dots,A_n) = (A_{\sigma^{-1}(1)},\dots,A_{\sigma^{-1}(n)})$.

1.2.2. $\nu_\sigma : R \to R$ is the k-algebra homomorphism satisfying

$\nu_\sigma(X_{i;pq}) = X_{\sigma(i);pq}$.

1.2.3. $\mu'_\sigma : \Sigma' \to \Sigma'$ is the k-algebra homomorphism such that

$\mu'_\sigma(X_i) = X_{\sigma(i)}$, $\mu'_\sigma(Y_i) = Y_{\sigma(i)}$, $\mu'_\sigma(Z_{ij}) = Z_{\sigma(i)\sigma(j)}$ and

$\mu'_\sigma(W_{ijk}) = W_{\sigma(i)\sigma(j)\sigma(k)}$.

1.2.4. $\mu_\sigma : \Sigma \to \Sigma$ is defined by $\mu_\sigma = \pi\mu'_\sigma j$ (cf. 1.1.)

1.2.5. $\overline{\mu}_\sigma : k^m \to k^m$ is the map such that for all $x \in k^m$:

$\overline{\mu}_\sigma(x) = (\dots\tilde{x}\mu_{\sigma^{-1}}(X_i)\dots,\dots\tilde{x}\mu_{\sigma^{-1}}(Y_i)\dots,\dots\tilde{x}\mu_{\sigma^{-1}}(Z_{ij})\dots,\dots\tilde{x}\mu_{\sigma^{-1}}(W_{ijk})\dots)$.

Notice that $\nu_\sigma : R \to R$ induces a unique morphism $\nu_\sigma^* : X \to X$ and that the restriction of this morphism to the set of closed points of X equals $\overline{\nu}_{\sigma^{-1}}$; μ_σ and $\overline{\mu}_{\sigma^{-1}}$ are connected in a similar way.

Using elementary properties of algebraic geometry and the definitions above it follows immediately that the diagrams below are commutative.

<u>1.3</u>.
$$\begin{tikzcd} R \arrow[r, "\tilde{A}"] & k \\ R \arrow[u, "\nu_\sigma^{-1}"] \arrow[ur, "\widetilde{\overline{\nu}_\sigma(A)}"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "\tilde{x}"] & k \\ \Sigma \arrow[u, "\mu_\sigma^{-1}"] \arrow[ur, "\widetilde{\overline{\mu}_\sigma(x)}"'] & \end{tikzcd} \qquad (A \in M(2,k)^n,\ x \in k^m)$$

<u>1.4</u>.
$$\begin{tikzcd} \Sigma' \arrow[r, "\mu'_\sigma"] \arrow[d, "\pi"'] & \Sigma' \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\mu_\sigma"'] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "\mu_\sigma"] \arrow[d, "\alpha"'] & \Sigma \arrow[d, "\alpha"] \\ R \arrow[r, "\nu_\sigma"'] & R \end{tikzcd} \qquad \begin{tikzcd} M(2,k)^n \arrow[r, "\overline{\nu}_\sigma"] \arrow[d, "i"'] & M(2,k)^n \arrow[d, "i"] \\ k^m \arrow[r, "\overline{\mu}_\sigma"'] & k^m \end{tikzcd}$$

<u>definition 1.5.1</u>. $\delta' : \Sigma' \to \Sigma'$ is the k-algebra homomorphism such that

(i) $\delta'(X_1) = X_1+X_2$; $\delta'(X_2) = -X_2$; $\delta'(X_i) = X_i$ if $i \geq 3$

(ii) $\delta'(Y_1) = Y_1+Y_2+2Z_{12}$; $\delta'(Y_i) = Y_i$ if $i \geq 2$

(iii) $\delta'(Z_{12}) = -Z_{12}-Y_2$; $\delta'(Z_{21}) = -Z_{21}-Y_2$

$\delta'(Z_{1j}) = Z_{1j}+Z_{2j}$; $\delta'(Z_{j1}) = Z_{j1}+Z_{j2}$ $\quad (j \geq 3)$

$\delta'(Z_{2j}) = -Z_{2j}$; $\delta'(Z_{j2}) = -Z_{j2}$ $\quad (j \geq 3)$

$\delta'(Z_{ij}) = Z_{ij}$ if $\{i,j\} \cap \{1,2\} = \emptyset$

(iv) $\delta'(W_{ijk}) = W_{ijk}$ if $\{i,j,k\} \cap \{1,2\} = \emptyset$

$\delta'(W_{ijk}) = -W_{ijk}$ if $2 \in \{i,j,k\} \cap \{1,2\}$

$$\left.\begin{array}{l} \delta'(W_{1ij}) = W_{1ij}+W_{2ij} \\ \delta'(W_{i1j}) = W_{i1j}+W_{i2j} \\ \delta'(W_{ij1}) = W_{ij1}+W_{ij2} \end{array}\right\} \text{ if } 2 \notin \{i,j\}$$

<u>definition 1.5.2</u>. Define $\delta : \Sigma \to \Sigma$ by $\delta = \pi\delta' j$. Explicitly:

(i) $\delta(X_1) = X_1+X_2$; $\delta(X_2) = -X_2$; $\delta(X_i) = X_i$ if $i \geq 3$.

(ii) $\delta(Y_1) = Y_1+Y_2+2Z_{12}$; $\delta(Y_i) = Y_i$ if $i \geq 2$.

(iii) $\delta(Z_{12}) = -Z_{12}-Y_2$

$\delta(Z_{1j}) = Z_{1j}+Z_{2j}$ if $j \geq 3$

$$\delta(Z_{2j}) = -Z_{2j} \quad \text{if } j \geq 3$$

$$\delta(Z_{ij}) = Z_{ij} \quad \text{if } 3 \leq i < j \leq n$$

(iv) $\delta(W_{12k}) = -W_{12k}$

$$\delta(W_{1jk}) = W_{1jk}+W_{2jk} \quad \text{if } 3 \leq j < k \leq n$$

$$\delta(W_{2jk}) = -W_{2jk} \quad \text{if } 3 \leq j < k \leq n$$

$$\delta(W_{ijk}) = W_{ijk} \quad \text{if } 3 \leq i < j < k \leq n$$

<u>definition 1.5.3</u>. $d : R \to R$ is the k-algebra homomorphism satisfying:

(i) $d(X_{1;pq}) = X_{1;pq}+X_{2;pq}$

(ii) $d(X_{2;pq}) = -X_{2;pq}$ $\qquad (p,q) \in \{(1,1),(1,2),(2,1),(2,2)\}$

(iii) $d(X_{i;pq}) = X_{i;pq}$ if $i \geq 3$

<u>definition 1.5.4</u>. $\bar{d} : M(2,k)^n \to M(2,k)^n$ is the map such that $\bar{d}(A_1,\ldots,A_n) = (A_1+A_2,-A_2,A_3,\ldots,A_n)$ for all $(A_1,\ldots,A_n) \in M(2,k)^n$.

<u>definition 1.5.5</u>. $\bar{\delta} : k^m \to k^m$ is the map such that for all $x \in k^m$:
$\bar{\delta}(x) = (\ldots\tilde{x}\delta(X_i)\ldots,\ldots\tilde{x}\delta(Y_i)\ldots,\ldots\tilde{x}\delta(Z_{ij})\ldots,\ldots\tilde{x}\delta(W_{ijk})\ldots)$

<u>remark</u>: (1) $d : R \to R$ induces a unique morphism $d^* : \operatorname{Spec} R \to \operatorname{Spec} R$; $\bar{d}$ equals the restriction of d^* to the set of closed points of Spec R. $\delta : \Sigma \to \Sigma$ and $\bar{\delta} : k^m \to k^m$ are connected in a similar way.

(2) $\delta', \delta, \bar{\delta}, d$ and $\bar{d}$ are involutions (i.e., their square is the identity map).

(3) The diagrams below are commutative.

<u>1.6</u>. $$\begin{tikzcd} \Sigma \arrow[r, "\tilde{x}"] & k \\ \Sigma \arrow[u, "\delta"] \arrow[ur, "\widetilde{\bar{\delta}(x)}"'] & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\tilde{A}"] & k \\ R \arrow[u, "d"] \arrow[ur, "\widetilde{\bar{d}(A)}"'] & \end{tikzcd} \qquad (x \in k^m,\ A \in M(2,k)^n)$$

<u>1.7</u>.
$$\begin{tikzcd} \Sigma' \arrow[r, "\delta'"] \arrow[d, "\pi"'] & \Sigma' \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\delta"] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "\delta"] \arrow[d, "\alpha"'] & \Sigma \arrow[d, "\alpha"] \\ R \arrow[r, "d"] & R \end{tikzcd} \qquad \begin{tikzcd} M(2,k)^n \arrow[r, "\bar{d}"] \arrow[d, "i"'] & M(2,k)^n \arrow[d, "i"] \\ k^m \arrow[r, "\bar{\delta}"] & k^m \end{tikzcd}$$

Next we define an ideal J in Σ which will appear to be identical to the kernel of $\alpha : \Sigma \to R$.

<u>definition 1.8</u>. $J = \pi J'$ where J' is the ideal in Σ' generated by:

(a) $f(i_1,i_2,i_3)=4W^2_{i_1i_2i_3}-Y_{i_1}Z^2_{i_2i_3}-Y_{i_2}Z^2_{i_1i_3}-Y_{i_3}Z^2_{i_1i_2}+2Z_{i_1i_2}Z_{i_1i_3}Z_{i_2i_3}+Y_{i_1}Y_{i_2}$
where i_1,i_2 and i_3 are three different integers between 1 and n.

(b) $g(i_1,i_2,i_3,i_4)=-Y_{i_1}W_{i_2i_3i_4}+Z_{i_1i_2}W_{i_1i_3i_4}-Z_{i_1i_3}W_{i_1i_2i_4}+Z_{i_1i_4}W_{i_1i_2i_3}$,
where $i_1,\ldots,i_4$ are four different integers between 1 and n.

(c) $h(i_1,i_2,i_3,i_4)=4W_{i_1i_2i_3}W_{i_2i_3i_4}-Z_{i_1i_4}(Z^2_{i_2i_3}-Y_{i_2}Y_{i_3})+$
$+Z_{i_1i_3}(Z_{i_2i_3}Z_{i_2i_4}-Y_{i_2}Z_{i_3i_4})+Z_{i_1i_2}(Z_{i_2i_3}Z_{i_3i_4}-Y_{i_3}Z_{i_2i_4})$,
where $i_1,\ldots,i_4$ are four different integers between 1 and n.

(d) $k(i_1,i_2,i_3,i_4,i_5)=4W_{i_1i_2i_3}W_{i_1i_4i_5}+Y_{i_1}(Z_{i_2i_4}Z_{i_3i_5}-Z_{i_2i_5}Z_{i_3i_4})+$
$+Z_{i_1i_3}(Z_{i_1i_4}Z_{i_2i_5}-Z_{i_2i_4}Z_{i_1i_5})+$
$+Z_{i_1i_2}(Z_{i_1i_5}Z_{i_3i_4}-Z_{i_1i_4}Z_{i_3i_5})$,
where $i_1,\ldots,i_5$ are five different integers between 1 and n.

(e) $l(i_1,i_2,i_3,i_4,i_5)=Z_{i_4i_5}W_{i_1i_2i_3}+Z_{i_2i_5}W_{i_1i_3i_4}+Z_{i_2i_3}W_{i_1i_4i_5}-Z_{i_3i_4}W_{i_1i_2i_5}$
$-Z_{i_1i_4}W_{i_2i_3i_5}-Z_{i_1i_2}W_{i_3i_4i_5}$,
where $i_1,\ldots,i_5$ are five different integers between 1 and n.

(f) $m(i_1,i_2,i_3,i_4,i_5,i_6)=4W_{i_1i_2i_3}W_{i_4i_5i_6}+4W_{i_1i_3i_4}W_{i_2i_5i_6}+$
$+Z_{i_1i_2}(Z_{i_3i_5}Z_{i_4i_6}-Z_{i_3i_6}Z_{i_4i_5})+Z_{i_1i_4}(Z_{i_2i_5}Z_{i_3i_6}-Z_{i_3i_5}Z_{i_2i_6})+$
$+Z_{i_1i_5}(Z_{i_2i_6}Z_{i_3i_4}-Z_{i_2i_3}Z_{i_4i_6})+Z_{i_1i_6}(Z_{i_2i_3}Z_{i_4i_5}-Z_{i_2i_5}Z_{i_3i_4})$,
where $i_1,\ldots,i_6$ are six different integers between 1 and n.

<u>remark</u>: In case $n \geq 6$ we have the full scale of elements defined above at our disposal. In case $n = 3$, for instance, J' is meant to be the ideal generated by the elements listed under (a), etc.

<u>lemma 1.9</u>. If $(A,B,C) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \begin{pmatrix} i & j \\ k & l \end{pmatrix}) \in M(2,k)^3$ then

$2TrAB-TrATrB = (a-d)(e-h)+2(bg+cf)$

$TrABC-TrACB = (a-d)(fk-gj)-(e-h)(bk-cj)+(i-l)(bg-cf)$.

<u>proof</u>: Just compute.

<u>theorem 1.10</u>. $J \subset \ker \alpha$.

<u>proof</u>: Assume $m(1,2,3,4,5,6) \in \ker \alpha\pi$. Then also $m(i_1,i_2,i_3,i_4,i_5,i_6) \in \ker \alpha\pi$ because there exists a $\sigma \in S_n$ such that $\sigma(p) = i_p$ $(1 \leq p \leq 6)$ and $\alpha\pi m(i_1,\ldots,i_6) = \alpha\pi m(\sigma(1),\ldots,\sigma(6)) =$ $= \alpha\pi\mu'_\sigma m(1,2,\ldots,6) = \alpha\mu_\sigma\pi m(1,\ldots,6) = \nu_\sigma\alpha\pi m(1,\ldots,6) = 0$ (cf. 1.2 and 1.4.) Reasoning in this way we may restrict ourselves to proving that the following six elements belong to $\ker \alpha\pi$:

$m(1,2,3,4,5,6)$; $l(1,2,3,4,5)$; $k(1,2,3,4,5)$; $h(1,2,3,4)$; $g(1,2,3,4)$ and $f(1,2,3)$.

I claim that it is sufficient to prove that $m(1,2,3,4,5,6)$ and $l(1,2,3,4,5)$ belong to $\ker \alpha\pi$. This can be seen as follows:

$f(1,2,3) \in \ker \alpha\pi \leftrightarrow 0=(\alpha\pi f(1,2,3))(A_1,\ldots,A_n)$ for all $(A_1,\ldots,A_n) \in M(2,k)^n$.

But $(\alpha\pi f(1,2,3))(A_1,\ldots,A_n)=(\alpha\pi m(1,2,3,4,5,6))(A_1,A_2,A_3,A_1,A_2,A_3,0,\ldots,0)=0$.

$g(1,2,3,4) \in \ker \alpha\pi$ because

$(\alpha\pi g(1,2,3,4))(A_1,\ldots,A_n)=(\alpha\pi l(1,\ldots,5))(A_1,A_2,A_3,A_1,A_4,0,\ldots,0) = 0$

$h(1,2,3,4) \in \ker \alpha\pi$ because

$(\alpha\pi h(1,2,3,4))(A_1,\ldots,A_n)=(\alpha\pi m(1,\ldots,6))(A_1,A_2,A_3,A_3,A_4,A_2,0,\ldots,0) = 0$

$k(1,2,3,4,5) \in \ker \alpha\pi$ because

$(\alpha\pi k(1,\ldots,5))(A_1,\ldots,A_n)=(\alpha\pi m(1,\ldots,6))(A_1,A_2,A_3,A_1,A_4,A_5,0,\ldots,0) = 0$

Introducing $Z_{ii} = Y_i$ and $W_{ijk} = 0$ if $\#\{i,j,k\} \neq 3$ the statements above can be written in the form: $\pi f(1,2,3) = \pi m(1,2,3,1,2,3)$

$$\pi g(1,2,3,4) = \pi l(1,2,3,1,4)$$

$$\pi h(1,2,3,4) = \pi m(1,2,3,3,4,2)$$

$$\pi k(1,2,3,4,5) = \pi m(1,2,3,1,4,5).$$

It remains to prove that the following elements belong to ker $\alpha\pi$:

$l(1,2,3,4,5) = Z_{45}W_{123}+Z_{25}W_{134}+Z_{23}W_{145}-Z_{34}W_{125}-Z_{14}W_{235}-Z_{12}W_{345}$ and

$$m(1,2,3,4,5,6) = 4W_{123}W_{456}+4W_{134}W_{256}+Z_{12}(Z_{35}Z_{46}-Z_{36}Z_{45})+$$
$$+Z_{14}(Z_{25}Z_{36}-Z_{35}Z_{26})+Z_{15}(Z_{26}Z_{34}-Z_{23}Z_{46})+Z_{16}(Z_{23}Z_{45}-Z_{25}Z_{34}).$$

Let $(A_1,\dots,A_n) \in M(2,k)^n$ where

$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $A_3 = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$, $A_4 = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $A_5 = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ and

$A_6 = \begin{pmatrix} v & w \\ x & z \end{pmatrix}$.

Then: $(\alpha\pi l(1,2,3,4,5))(A_1,\dots,A_n)$

$$\|$$

$$[(m-q)(r-u)+2(nt+ps)][(a-d)(fk-gj)-(e-h)(bk-cj)+(i-l)(bg-cf)]+$$
$$+[(e-h)(r-u)+2(ft+gs)][(a-d)(jp-kn)-(i-l)(bp-cn)+(m-q)(bk-cj)]+$$
$$+[(e-h)(i-l)+2(fk+gj)][(a-d)(nt-ps)-(m-q)(bt-cs)+(r-u)(bp-cn)]+$$
$$-[(i-l)(m-q)+2(jp+kn)][(a-d)(ft-gs)-(e-h)(bt-cs)+(r-u)(bg-cf)]+$$
$$-[(a-d)(m-q)+2(bp+cn)][(e-h)(jt-ks)-(i-l)(ft-gs)+(r-u)(fk-gj)]+$$
$$-[(a-d)(e-h)+2(bg+cf)][(i-l)(nt-ps)-(m-q)(jt-ks)+(r-u)(jp-kn)]$$

$$\|$$

$$(a-d)(e-h)(i-l)[nt-ps-(nt-ps)]+(a-d)(e-h)(m-q)[-(jt-ks)+jt-ks]+$$
$$+(a-d)(e-h)(r-u)[jp-kn-(jp-kn)]+(a-d)(i-l)(m-q)[-(ft-gs)+ft-gs]+$$
$$+(a-d)(m-q)(r-u)[fk-gj-(fk-gj)]+(e-h)(i-l)(m-q)[-(bt-cs)+bt-cs]+$$
$$+(e-h)(i-l)(r-u)[-(bp-cn)+bp-cn]+(e-h)(m-q)(r-u)[-(bk-cj)+(bk-cj)]+$$
$$+(i-l)(m-q)(r-u)[bg-cf-(bg-cf)]+$$

$$+$$

$$+$$

$$+2(a-d)[(nt+ps)(fk-gj)+(ft+gs)(jp-kn)+(fk+gj)(nt-ps)-(jp+kn)(ft-gs)]+$$

$$+2(e-h)[-(nt+ps)(bk-cj)+(jp+kn)(bt-cs)-(bp+cn)(jt-ks)]+$$

$$+2(i-l)[(nt+ps)(bg-cf)-(ft+gs)(bp-cn)+(bp+cn)(ft-gs)-(bg+cf)(nt-ps)]+$$

$$+2(m-q)[(ft+gs)(bk-cj)-(fk+gj)(bt-cs)+(bg+cf)(jt-ks)]+$$

$$+2(r-u)[(fk+gj)(bp-cn)-(jp+kn)(bg-cf)-(bp+cn)(fk-gj)-(bg+cf)(jp-kn)].$$

$$\|$$

$$0$$

$\rightarrow \alpha\pi l(1,2,3,4,5) = 0 \rightarrow l(1,2,3,4,5) \in \ker \alpha\pi.$

$$(\alpha\pi m(1,2,3,4,5,6))(A_1,\ldots,A_n)$$

$$\|$$

$$4[(a-d)(fk-gj)-(e-h)(bk-cj)+(i-l)(bg-cf)][(m-q)(sx-tw)-(r-u)(nx-pw)+(v-z)(nt-ps)]+$$

$$+4[(a-d)(jp-kn)-(i-l)(bp-cn)+(m-q)(bk-cj)][(e-h)(sx-tw)-(r-u)(fx-gw)++(v-z)(ft-gs)]+$$

$$+\ [(a-d)(e-h)+2(bg+cf)][(i-l)(r-u)+2(jt+ks)][(m-q)(v-z)+2(nx+pw)]+$$

$$-\ [(a-d)(e-h)+2(bg+cf)][(i-l)(v-z)+2(jx+kw)][(m-q)(r-u)+2(nt+ps)]+$$

$$+\ [(a-d)(m-q)+2(bp+cn)][(e-h)(r-u)+2(ft+gs)][(i-l)(v-z)+2(jx+kw)]+$$

$$-\ [(a-d)(m-q)+2(bp+cn)][(i-l)(r-u)+2(jt+ks)][(e-h)(v-z)+2(fx+gw)]+$$

$$+\ [(a-d)(r-u)+2(bt+cs)][(e-h)(v-z)+2(fx+gw)][(i-l)(m-q)+2(jp+kn)]+$$

$$-\ [(a-d)(r-u)+2(bt+cs)][(e-h)(i-l)+2(fk+gj)][(m-q)(v-z)+2(nx+pw)]+$$

$$+\ [(a-d)(v-z)+2(bx+cw)][(e-h)(i-l)+2(fk+gj)][(m-q)(r-u)+2(nt+ps)]+$$

$$-\ [(a-d)(v-z)+2(bx+cw)][(e-h)(r-u)+2(ft+gs)][(i-l)(m-q)+2(jp+kn)].$$

$$\|$$

$$\Vert$$

$$4(a-d)(e-h)[(jp-kn)(sx-tw)+(jt+ks)(nx+pw)-(jx+kw)(nt+ps)]+$$

$$+4(a-d)(m-q)[(fk-gj)(sx-tw)+(ft+gs)(jx+kw)-(jt+ks)(fx+gw)]+$$

$$+4(a-d)(r-u)[-(fk-gj)(nx-pw)-(jp-kn)(fx-gw)+(fx+gw)(jp+kn)-(fk+gj)(nx+pw)]+$$

$$+4(a-d)(v-z)[(fk-gj)(nt-ps)+(jp-kn)(ft-gs)+(fk+gj)(nt+ps)-(ft+gs)(jp+kn)]+$$

$$+4(e-h)(i-l)[-(bp-cn)(sx-tw)-(bt+cs)(nx+pw)+(bx+cw)(nt+ps)]+$$

$$+4(e-h)(m-q)[-(bk-cj)(sx-tw)+(bk-cj)(sx-tw)]+$$

$$+4(e-h)(r-u)[(bk-cj)(nx-pw)+(bp+cn)(jx+kw)-(bx+cw)(jp+kn)]+$$

$$+4(e-h)(v-z)[(-(bk-cj)(nt-ps)-(bp+cn)(jt+ks)+(bt+cs)(jp+kn)]+$$

$$+4(i-l)(m-q)[(bg-cf)(sx-tw)+(bt+cs)(fx+gw)-(bx+cw)(ft+gs)]+$$

$$+4(i-l)(r-u)[-(bg-cf)(nx-pw)+(bp-cn)(fx-gw)+(bg+cf)(nx+pw)-(bp+cn)(fx+gw)]+$$

$$+4(i-l)(v-z)[(bg-cf)(nt-ps)-(bp-cn)(ft-gs)-(bg+cf)(nt+ps)+(bp+cn)(ft+gs)]+$$

$$+4(m-q)(r-u)[-(bk-cj)(fx-gw)-(bg+cf)(jx+kw)+(bx+cw)(fk+gj)]+$$

$$+4(m-q)(v-z)[(bk-cj)(ft-gs)+(bg+cf)(jt+ks)-(bt+cs)(fk+gj)]+$$

$$+$$

$$+2(a-d)(e-h)(i-l)(r-u)[nx+pw-(nx+pw)]+$$

$$+2(a-d)(e-h)(i-l)(v-z)[-(nt+ps)+nt+ps]+$$

$$+2(a-d)(e-h)(m-q)(r-u)[-(jx+kw)+jx+kw]+$$

$$+2(a-d)(e-h)(m-q)(v-z)[jt+ks-(jt+ks)]+$$

$$+2(a-d)(e-h)(r-u)(v-z)[jp+kn-(jp+kn)]+$$

$$+2(a-d)(i-l)(m-q)(r-u)[-(fx+gw)+fx+gw]+$$

$$+2(a-d)(i-l)(m-q)(v-z)[ft+gs-(ft+gs)]+$$

$$+2(a-d)(m-q)(r-u)(v-z)[-(fk+gj)+fk+gj]+$$

$$+$$

$$+8(bg+cf)[(jt+ks)(nx+pw)-(jx+kw)(nt+ps)]+$$

$$+8(bp+cn)[(ft+gs)(jx+kw)-(jt+ks)(fx+gw)]+$$

$$+8(bt+cs)[(fx+gw)(jp+kn)-(fk+gj)(nx+pw)]+$$

$$+8(bx+cw)[(fk+gj)(nt+ps)-(jp+kn)(ft+gs)].$$

$$\Vert$$

$$0$$

$$\to \alpha\pi m(1,2,3,4,5,6) = 0 \to m(1,2,3,4,5,6) \in \ker \alpha\pi .$$

remark 1.11. The connection with Procesi's results (cf. I, §1).

Define $\tau : {}^{T}n/\mathcal{Q} \to R$ by $\tau(\mathrm{Tr}X_{i_1}\dots X_{i_r} + \mathcal{Q}) = \mathrm{Tr}\underline{X}_{i_1}\dots\underline{X}_{i_r}$, then $\alpha = \tau\omega$ so we have the commutative diagram:

$$\begin{tikzcd} \Sigma' \arrow[r, "\pi"] \arrow[drr, "\alpha"'] & \Sigma \arrow[r, "\omega"] & {}^{T}n/\mathcal{Q} \arrow[d, "\tau"] \\ & & R \end{tikzcd}$$

As above: $Z_{ii} = Y_i$ and $W_{ijk} = 0$ if $\#\{i,j,k\} \neq 3$. Now define for each subset $\{i_1,\dots,i_6\}$ of $\{1,\dots,n\}$, where $i_1,\dots,i_6$ need not be different, the element $\theta(i_1,i_2,i_3,i_4,i_5,i_6)$ of Σ' by:

$\theta(i_1,\dots,i_6) = m(i_1,\dots,i_6)+2X_{i_5}l(i_1,i_2,i_3,i_4,i_6)+2X_{i_6}l(i_1,i_2,i_3,i_4,i_5)$.

Then, with some perseverance it can be proved that

$$\omega\pi(\theta(i_1,\dots,i_6)) = 32F(X_{i_1}X_{i_2},X_{i_3}X_{i_4},X_{i_5}X_{i_6})+\mathcal{Q} = 0$$

Therefore $\pi\theta(i_1,\dots,i_6)$ belongs to ker α. Now we have:

(1) $\pi f(i_1,i_2,i_3) = \pi\theta(i_1,i_2,i_3,i_1,i_2,i_3) \in \ker\alpha$

(2) $\left.\begin{array}{l} 2X_{i_1}\pi g(i_1,i_2,i_3,i_4)=\pi\theta(i_1,i_2,i_3,i_1,i_4,i_1)\in\ker\alpha \\ X_{i_1} \notin \ker\alpha;\ \ker\alpha \text{ is prime} \end{array}\right\} \to \pi g(i_1,i_2,i_3,i_4)\in\ker\alpha$

(3) $\pi h(i_1,i_2,i_3,i_4)=\frac{1}{2}\pi[\theta(i_1,i_2,i_3,i_4,i_2,i_3)-\theta(i_1,i_2,i_3,i_4,i_3,i_2)]\in\ker\alpha$

(4) $\pi k(i_1,i_2,i_3,i_4,i_5)=\pi[\theta(i_1,i_1,i_2,i_3,i_4,i_5)-2X_{i_4}g(i_1,i_2,i_3,i_5)+$
$-2X_{i_5}g(i_1,i_2,i_3,i_4)]\in\ker\alpha$

(5) $2X_{i_1}\pi l(i_1,i_2,i_3,i_4,i_5)=\pi[\theta(i_1,i_2,i_3,i_4,i_5,i_1)-k(i_1,i_2,i_3,i_4,i_5)+$
$-k(i_1,i_3,i_4,i_2,i_5)]\in\ker\alpha$

(6) $\pi m(i_1,i_2,i_3,i_4,i_5,i_6)=\pi[\theta(i_1,i_2,i_3,i_4,i_5,i_6)-2X_{i_5}l(i_1,i_2,i_3,i_4,i_6)+$
$-2X_{i_6}l(i_1,i_2,i_3,i_4,i_5)]\in\ker\alpha$.

§2. $\underline{J = \ker \alpha}$.

<u>lemma 2.1</u>. $\ker \alpha = \sqrt{J}$ if for each $x \in V(J) \subset k^m$ there exists $A \in M(2,k)^n$ such that $i(A) = x$.

<u>proof</u>: We already know $J \subset \ker \alpha$. Because $\ker \alpha$ is prime we also have $\sqrt{J} \subset \ker \alpha$. It remains to prove that $\ker \alpha \subset \sqrt{J}$. Now consider the equivalences: $t \in \ker \alpha \Leftrightarrow \tilde{A}(\alpha(t)) = 0$ for all $A \in M(2,k)^n$

$$\Leftrightarrow \widetilde{i(A)}(t) = 0 \text{ for all } A \in M(2,k)^n \text{ (cf. I.1.10)}$$

$$\Leftrightarrow \tilde{x}(t) = 0 \text{ for all } x \in V(J)$$

$$\Leftrightarrow t \in \sqrt{J} \text{ (Nullstellensatz).}$$

<u>definition 2.2</u>. For all $1 \leq i < j < k \leq n$ we write f_{ijk} instead of $\pi f(i,j,k)$.

$V(\Delta f_{ijk}) \subset k^m$ denotes the set of common zeros of the partial derivatives of f_{ijk} (cf. III, §2).

<u>definition 2.3</u>. K is the subideal of J generated by $\pi f(1,2,k)$ $(3 \leq k \leq n)$, $\pi g(1,i,j,k)$ $(2 \leq i < j < k \leq n)$, $\pi g(2,1,i,j)$ $(3 \leq i < j \leq n)$ and $\pi h(i,1,2,j)$ $(3 \leq i < j \leq n)$.

To simplify the reading we first give the framework of this paragraph.

th.2.16. $\ker \alpha = J$

$\uparrow$ using lemma 2.14 and 2.15

$\ker \alpha = \sqrt{J}$

$\uparrow$ using lemma 2.1.

If $x \in V(J)$ then there exists $A \in M(2,k)^n$ such that $i(A) = x$.

$\uparrow$ $\qquad$ $\uparrow$

th.2.13. $x \in V(J)$ and $x \notin V(\Delta f_{ijk})$ for some $1 \leq i < j < k \leq n$ iff there exists an irreducible element A in $M(2,k)^n$ such that $i(A) = x$.

lemma 2.4. $\bigcap_{1 \leq i < j < k \leq n} V(\Delta f_{ijk}) \subset V(J)$.

th.2.5. $x \in V(\Delta f_{ijk})$ for all $1 \leq i < j < k \leq n$ iff there exists a diagonalizable $A \in M(2,k)^n$ such that $i(A) = x$.

$\uparrow$

prop.2.12. If $x \in V(J) - V(\Delta f_{123})$ then there exists $A \in M(2,k)^n$ such that $i(A) = x$.

$\uparrow$ using lemma 2.11.

prop.2.10. $J_0 = K + \mu_{(12)}K + \delta K \subset J$. If $x \in V(J_0) - V(Z_{12}^2 - Y_1 Y_2)$ then there exists $A \in M(2,k)^n$ such that $i(A) = x$.

$\uparrow$ using lemma 2.7, 2.8 and 2.9.

th.2.6. If $x \in V(K)$, $x \notin V(Z_{12}^2 - Y_1 Y_2) \cup V(Y_1)$ then $x = i(A)$ for some $A \in M(2,k)^n$.

---

lemma 2.4. $\bigcap_{1 \leq i < j < k \leq n} V(\Delta f_{ijk}) = V(R) \subset V(J)$ where R is the ideal in Σ generated by:

W_{ijk}	$(1 \leq i < j < k \leq n)$
$Z_{ij}^2 - Y_i Y_j$	$(1 \leq i < j \leq n)$

$$\left.\begin{array}{l} Z_{ij}Z_{ik}-Y_iZ_{jk} \\ Z_{ij}Z_{jk}-Y_jZ_{ik} \\ Z_{ik}Z_{jk}-Y_kZ_{ij} \end{array}\right\} \qquad (1 \le i < j < k \le n)$$

<u>proof</u>:

$$V(\Delta f_{ijk}) = V(W_{ijk}, Z_{ij}^2-Y_iY_j, Z_{ik}^2-Y_iY_k, Z_{jk}^2-Y_jY_k, Z_{ij}Z_{ik}-Y_iZ_{jk}, Z_{ij}Z_{jk}-Y_jZ_{ik}, Z_{ik}Z_{jk}-Y_kZ_{ij}).$$

Consequently we have $\bigcap_{1\le i<j<k\le n} V(\Delta f_{ijk}) = V(R)$.

It remains to prove $V(R) \subset V(J)$.

(a) $f(i_1,i_2,i_3) = 4W^2_{i_1i_2i_3} - Y_{i_3}(Z^2_{i_1i_2} - Y_{i_1}Y_{i_2}) - Y_{i_2}(Z^2_{i_1i_3} - Y_{i_1}Y_{i_3}) + Y_{i_1}(Z^2_{i_2i_3} - Y_{i_2}Y_{i_3}) + 2Z_{i_2i_3}(Z_{i_1i_2}Z_{i_1i_3} - Y_{i_1}Z_{i_2i_3})$.

Therefore $\pi f(i_1,i_2,i_3) \in R$.

(b) From the definition of J (IV.1.8.) and R it follows immediately:

$$\pi g(i_1,i_2,i_3,i_4) \in R$$
$$\pi h(i_1,i_2,i_3,i_4) \in R$$
$$\pi l(i_1,i_2,i_3,i_4,i_5) \in R$$

(c) We claim that $\pi k(i_1,i_2,i_3,i_4,i_5)$ and $\pi m(i_1,i_2,i_3,i_4,i_5,i_6)$ belong to $\sqrt{R}$.

Clearly this is true if $\pi(Z_{ij}Z_{kl}-Z_{pq}Z_{rs}) \in \sqrt{R}$ in case $\{i,j,k,l\} = \{p,q,r,s\}$. In view of the definition of π, we can restrict ourselves to the following two possibilities:

(i) $p = i$, $q = k$, $r = j$, $s = l$. Then:

$$\begin{aligned}\pi(Z_{ij}Z_{kl}-Z_{ik}Z_{jl})^2 &= \pi(Z^2_{ij}Z^2_{kl}+Z^2_{ik}Z^2_{jl}-2Z_{ij}Z_{ik}Z_{kl}Z_{jl}) = \\ &= \pi[Z^2_{kl}(Z^2_{ij}-Y_iY_j)+Z^2_{ik}(Z^2_{jl}-Y_jY_l)-Y_iZ_{kl}(Z_{jk}Z_{jl}-Y_jZ_{kl})+ \\ &\quad -Y_lZ_{ik}(Z_{ij}Z_{jk}-Y_jZ_{ik})-Z_{jl}Z_{kl}(Z_{ij}Z_{ik}-Y_iZ_{jk})+ \\ &\quad -Z_{ij}Z_{ik}(Z_{jl}Z_{kl}-Y_lZ_{jk})] \in R.\end{aligned}$$

(ii) $p = i$, $q = l$, $r = j$, $s = k$. Then:

$$\pi(Z_{ij}Z_{kl}-Z_{il}Z_{jk})^2 = \pi(Z_{ij}^2Z_{kl}^2+Z_{il}^2Z_{jk}^2-2Z_{ij}Z_{il}Z_{jk}Z_{kl}) =$$
$$= \pi[Z_{ij}^2(Z_{kl}^2-Y_kY_l)-Y_kZ_{ij}(Z_{il}Z_{jl}-Y_lZ_{ij})-Z_{ij}Z_{il}(Z_{jk}Z_{kl}-Y_kZ_{jl})+$$
$$+Z_{jk}^2(Z_{il}^2-Y_iY_l)-Y_iZ_{jk}(Z_{jl}Z_{kl}-Y_lZ_{jk})-Z_{jk}Z_{kl}(Z_{ij}Z_{il}-Y_iZ_{jl})] \in R.$$

From (a), (b) and (c) above it follows that $J \subset \sqrt{R}$ which, in turn, implies $V(J) \supset V(\sqrt{R}) = V(R)$.

<u>theorem 2.5</u>. $x \in \bigcap_{1\le i<j<k\le n} V(\Delta f_{ijk})$ iff there exists a diagonalizable element A in $M(2,k)^n$ such that $i(A) = x$.

<u>proof</u>:

(i) If $A = (A_1,\dots,A_n)$ is diagonalizable, A is particularly reducible. Using II.1.6. this implies that (A_i,A_j,A_k) is reducible for all $1 \le i < j < k \le n$, which in view of th.III.2.3 is equivalent to $i(A_1,\dots,A_n) \in V(\Delta f_{ijk})$ for all $1 \le i < j < k \le n$. So, incidentally, we proved: <u>$A \in M(2,k)^n$ is reducible iff $i(A) \in V(R)$</u>.

(ii) Let $x = (x_1,\dots,x_n;y_1,\dots,y_n;z_{12},\dots,z_{n-1,n};0,\dots,0) \in V(R) = \bigcap_{1\le i<j<k\le n} V(\Delta f_{ijk})$. For every $1 \le i \le n$ we choose an element $\omega_i \in k$ such that $\omega_i^2 = y_i$. Define for all $1 \le i \le n$: $A_i = \begin{pmatrix} \xi_i & 0 \\ 0 & x_i-\xi_i \end{pmatrix}$, $\eta_i = x_i-2\xi_i$. We ask for $\eta_1,\dots,\eta_n \in k$ such that $i(A_1,\dots,A_n) = x$.

Now $i(A_1,\dots,A_n) = x$ is equivalent to:

(1): $x_i = \mathrm{Tr}A_i$

(2): $y_i = 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 = 2[\xi_i^2+(x_i-\xi_i)^2]-x_i^2 = (x_i-2\xi_i)^2 = \eta_i^2$

(3): $z_{ij} = 2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j = 2[\xi_i\xi_j+(x_i-\xi_i)(x_j-\xi_j)]-x_ix_j = \eta_i\eta_j$

Therefore we have: $i(A_1,\dots,A_n) = x \Leftrightarrow \begin{cases} \eta_i^2 = \omega_i^2 \ (1\le i\le n) \text{ and} \\ \eta_i\eta_j = z_{ij} \ (1 \le i < j \le n) \end{cases}$

Distinguish two cases:

<u>case 1</u>: $\omega_1 = \omega_2 = \ldots = \omega_n = 0$.

Then also $y_1 = \ldots = y_n = 0$. But $x \in V(Z_{ij}^2 - Y_iY_j)$, so moreover $z_{ij} = 0$ for all $1 \leq i < j \leq n$. Taking $\eta_1 = \ldots = \eta_n = 0$ we then conclude $i(A_1,\ldots,A_n) = x$.

<u>case 2</u>: There exists $i_0 \in \{i,\ldots,n\}$ such that $\omega_{i_0} \neq 0$.

In this case we define $\eta_{i_0} = \omega_{i_0}$, $\eta_j = z_{ji_0}\omega_{i_0}^{-1}$ if $j < i_0$ and $\eta_j = z_{i_0j}\omega_{i_0}^{-1}$ if $j > i_0$. Because x belongs to $V(R)$ we have:

(a) $j < i_0 \rightarrow \eta_j^2 = z_{ji_0}^2\omega_{i_0}^{-2} = z_{ji_0}^2 y_{i_0}^{-1} = y_jy_{i_0}y_{i_0}^{-1} = y_j = \omega_j^2$

$\quad \eta_{i_0}^2 = \omega_{i_0}^2$

$\quad j > i_0 \rightarrow \eta_j^2 = z_{i_0j}^2\omega_{i_0}^{-2} = \omega_j^2$

(b) $i < j < i_0 \rightarrow \eta_i\eta_j = z_{ii_0}z_{ji_0}\omega_{i_0}^{-2} = z_{ij}y_{i_0}y_{i_0}^{-1} = z_{ij}$

$\quad i < i_0 \rightarrow \eta_i\eta_{i_0} = z_{ii_0}$

$\quad i < i_0 < j \rightarrow \eta_i\eta_j = z_{ii_0}z_{i_0j}y_{i_0}^{-1} = z_{ij}y_{i_0}y_{i_0}^{-1} = z_{ij}$

$\quad i_0 < j \rightarrow \eta_{i_0}\eta_j = z_{i_0j}$

$\quad i_0 < i < j \rightarrow \eta_i\eta_j = z_{i_0i}z_{i_0j}\omega_{i_0}^{-2} = z_{ij}$.

From (a) and (b) it follows that $i(A_1,\ldots,A_n) = x$.

<u>theorem 2.6</u>. If $x \in V(K)$ and $x \notin V(Z_{12}^2 - Y_1Y_2) \cup V(Y_1)$ then $x = i(A)$ for some $A \in M(2,k)^n$.

<u>proof</u>: Let $x = (\ldots a_i \ldots, \ldots b_i \ldots, \ldots c_{ij} \ldots, \ldots d_{ijk} \ldots)$. Introduce $d_{121} = d_{122} = 0$, $c_{11} = b_1$, $c_{21} = c_{12}$ and $c_{22} = b_2$. Choose $\omega \in k$ such that $\omega^2 = c_{12}^2 - b_1b_2$ ($\neq 0$).

For all $1 \leq k \leq n$ we define the matrix A_k by

$$A_k = \tfrac{1}{2}\begin{pmatrix} a_k-2d_{12k}\omega^{-1} & b_1^{-1}[c_{1k}+\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})] \\ c_{1k}-\omega^{-1}(c_{12}c_{1k}-b_1c_{2k}) & a_k+2d_{12k}\omega^{-1} \end{pmatrix}$$

In particular: $A_1 = \tfrac{1}{2}\begin{pmatrix} a_1 & 1 \\ b_1 & a_1 \end{pmatrix}$ and $A_2 = \tfrac{1}{2}\begin{pmatrix} a_2 & b_1^{-1}(c_{12}+\omega) \\ c_{12}-\omega & a_2 \end{pmatrix}$

We claim $i(A_1,\dots,A_n) = x$.

(1) Evidently $\mathrm{Tr}A_k = a_k$ for all $1 \leq k \leq n$.

(2)
$$\begin{aligned} 2\mathrm{Tr}A_k^2-(\mathrm{Tr}A_k)^2 &= \tfrac{1}{2}[(a_k-2d_{12k}\omega^{-1})^2+2b_1^{-1}(c_{1k}^2-\omega^{-2}(c_{12}c_{1k}-b_1c_{2k})^2)+ \\ &\qquad +(a_k+2d_{12k}\omega^{-1})^2]-a_k^2 = \\ &= (b_1\omega^2)^{-1}[4d_{12k}^2b_1+\omega^2c_{1k}^2-c_{12}^2c_{1k}^2+2b_1c_{12}c_{1k}c_{2k}-b_1^2c_{2k}^2] = \\ &= \omega^{-2}[4d_{12k}^2-b_1c_{2k}^2-b_2c_{1k}^2+2c_{12}c_{1k}c_{2k}] \end{aligned}$$

So: $2\mathrm{Tr}A_1^2-(\mathrm{Tr}A_1)^2 = \omega^{-2}[-b_1c_{12}^2-b_2b_1^2+2b_1c_{12}^2] = b_1\omega^{-2}(c_{12}^2-b_1b_2) = b_1$,

$2\mathrm{Tr}A_2^2-(\mathrm{Tr}A_2)^2 = \omega^{-2}[-b_1b_2^2-b_2c_{12}^2+2b_2c_{12}^2] = b_2\omega^{-2}(c_{12}^2-b_1b_2) = b_2$, and

if $k \geq 3$ $\;2\mathrm{Tr}A_k^2-(\mathrm{Tr}A_k)^2 = \omega^{-2}[\tilde{x}\pi f(1,2,k)+b_kc_{12}^2-b_1b_2b_k] = b_k$.

(3) If $1 \leq k < l \leq n$ then: $4(2\mathrm{Tr}A_kA_l-\mathrm{Tr}A_k\mathrm{Tr}A_l) =$

$$\begin{aligned} &= 16\omega^{-2}d_{12k}d_{12l}+2b_1^{-1}[c_{1k}+\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})][c_{1l}-\omega^{-1}(c_{12}c_{1l}-b_1c_{2l})]+ \\ &\qquad +2b_1^{-1}[c_{1k}-\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})][c_{1l}+\omega^{-1}(c_{12}c_{1l}-b_1c_{2l})] = \\ &= 16\omega^{-2}d_{12k}d_{12l}+4b_1^{-1}\omega^{-2}[c_{1k}c_{1l}\omega^2-(c_{12}c_{1k}-b_1c_{2k})(c_{12}c_{1l}-b_1c_{2l})] = \\ &= 16\omega^{-2}d_{12k}d_{12l}+4\omega^{-2}[c_{12}c_{1k}c_{2l}+c_{12}c_{2k}c_{1l}-b_1c_{2k}c_{2l}-b_2c_{1k}c_{1l}] = \\ &= 4\omega^{-2}[4d_{12k}d_{12l}-c_{kl}(c_{12}^2-b_1b_2)+c_{2k}(c_{12}c_{1l}-b_1c_{2l})+c_{1k}(c_{12}c_{2l}-b_2c_{1l})]+4c_{kl}. \end{aligned}$$

$k = 1,\ l = 2 \Rightarrow 2\mathrm{Tr}A_1A_2-\mathrm{Tr}A_1\mathrm{Tr}A_2 = c_{12}$

$k = 1,\ l \geq 3 \Rightarrow 2\mathrm{Tr}A_1A_l-\mathrm{Tr}A_1\mathrm{Tr}A_l = c_{1l}$

$k = 2,\ l \geq 3 \Rightarrow 2\mathrm{Tr}A_2A_l-\mathrm{Tr}A_2\mathrm{Tr}A_l = c_{2l}$

$3 \leq k < l \leq n \Rightarrow 2\mathrm{Tr}A_kA_l-\mathrm{Tr}A_k\mathrm{Tr}A_l = \omega^{-2}[\tilde{x}\pi h(1,2,k,l)]+c_{kl} = c_{kl}$.

(4) In view of lemma 1.9. we have for all $1 \leq i < j < k \leq n$:

$$
\begin{aligned}
& 8(\mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j) \\
& \Vert \\
& -4d_{12i}b_1^{-1}\omega^{-1}([c_{ij}+\omega^{-1}(c_{12}c_{1j}-b_1c_{2j})][c_{1k}-\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})]+ \\
& \qquad -[c_{1j}-\omega^{-1}(c_{12}c_{1j}-b_1c_{2j})][c_{1k}+\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})])+ \\
& -4d_{12j}b_1^{-1}\omega^{-1}([c_{1i}-\omega^{-1}(c_{12}c_{1i}-b_1c_{2i})][c_{1k}+\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})]+ \\
& \qquad -[c_{1i}+\omega^{-1}(c_{12}c_{1i}-b_1c_{2i})][c_{1k}-\omega^{-1}(c_{12}c_{1k}-b_1c_{2k})])+ \\
& -4d_{12k}b_1^{-1}\omega^{-1}([c_{1i}+\omega^{-1}(c_{12}c_{1i}-b_1c_{2i})][c_{1j}-\omega^{-1}(c_{12}c_{1j}-b_1c_{2j})]+ \\
& \qquad -[c_{1i}-\omega^{-1}(c_{12}c_{1i}-b_1c_{2i})][c_{1j}+\omega^{-1}(c_{12}c_{1j}-b_1c_{2j})] \\
& \Vert \\
& 8b_1^{-1}\omega^{-2}\{d_{12i}[c_{1j}(c_{12}c_{1k}-b_1c_{2k})-c_{1k}(c_{12}c_{1j}-b_1c_{2j})]+ \\
& \qquad +d_{12j}[c_{1k}(c_{12}c_{1i}-b_1c_{2i})-c_{1i}(c_{12}c_{1k}-b_1c_{2k})]+ \\
& \qquad +d_{12k}[c_{1i}(c_{12}c_{1j}-b_1c_{2j})-c_{1j}(c_{12}c_{1i}-b_1c_{2i})]\} \\
& \Vert \\
& 8\omega^{-2}[d_{12i}(c_{1k}c_{2j}-c_{1j}c_{2k})+d_{12j}(c_{1i}c_{2k}-c_{1k}c_{2i})+d_{12k}(c_{1j}c_{2i}-c_{1i}c_{2j})].
\end{aligned}
$$

Therefore we have:

(i) $k \geq 3 \Rightarrow \mathrm{Tr}A_1A_2A_k-\mathrm{Tr}A_1A_kA_2 = \omega^{-2}d_{12k}(c_{12}^2-b_1b_2) = d_{12k}$.

(ii) If $3 \leq j < k \leq n$ then:

$$
\begin{aligned}
\mathrm{Tr}A_1A_jA_k-\mathrm{Tr}A_1A_kA_j &= \omega^{-2}[d_{12j}(b_1c_{2k}-c_{12}c_{1k})+d_{12k}(c_{12}c_{1j}-b_1c_{2j})] = \\
&= \omega^{-2}c_{12}(c_{1j}d_{12k}-c_{1k}d_{12j})+\omega^{-2}b_1(c_{2k}d_{12j}-c_{2j}d_{12k}) = \\
&= \omega^{-2}c_{12}(-\tilde{x}\pi g(1,2,j,k)-b_1d_{2jk}+c_{12}d_{1jk})+ \\
&\qquad +\omega^{-2}b_1(-\tilde{x}\pi g(2,1,j,k)-b_2d_{1jk}+c_{12}d_{2jk}) = \\
&= \omega^{-2}(c_{12}^2-b_1b_2)d_{1jk} = \\
&= d_{1jk}
\end{aligned}
$$

(iii) If $2 < j < k \leq n$ then:

$$\mathrm{Tr}A_2A_jA_k-\mathrm{Tr}A_2A_kA_j = \omega^{-2}[d_{12j}(c_{12}c_{2k}-c_{1k}b_2)+d_{12k}(c_{1j}b_2-c_{12}c_{2j})] =$$
$$= \omega^{-2}c_{12}(c_{2k}d_{12j}-c_{2j}d_{12k})+\omega^{-2}b_2(c_{1j}d_{12k}-c_{1k}d_{12j}) =$$
$$= \omega^{-2}c_{12}(-\tilde{x}\pi g(2,1,j,k)-b_2d_{1jk}+c_{12}d_{2jk})+$$
$$+\omega^{-2}b_2(-\tilde{x}\pi g(1,2,j,k)-b_1d_{2jk}+c_{12}d_{1jk}) =$$
$$= \omega^{-2}(c_{12}^2-b_1b_2)d_{2jk} =$$
$$= d_{2jk}$$

(iv) If $3 \leq i < j < k \leq n$ then:

$$-b_1\omega^2(d_{ijk}-\mathrm{Tr}A_iA_jA_k+\mathrm{Tr}A_iA_kA_j)$$
$$\|$$
$$-b_1[(c_{12}^2-b_1b_2)d_{ijk}-d_{12i}(c_{1k}c_{2j}-c_{1j}c_{2k})-d_{12j}(c_{1i}c_{2k}-c_{1k}c_{2i})+$$
$$-d_{12k}(c_{1j}c_{2i}-c_{1i}c_{2j})]$$
$$\|$$
$$-b_1c_{12}^2d_{1jk}-b_1b_2[\tilde{x}\pi g(1,i,j,k)-c_{1i}d_{1jk}+c_{1j}d_{1ik}-c_{1k}d_{1ij}]+$$
$$+b_1d_{12i}(c_{1k}c_{2j}-c_{1j}c_{2k})+b_1d_{12j}(c_{1i}c_{2k}-c_{1k}c_{2i})+b_1d_{12k}(c_{1j}c_{2i}-c_{1i}c_{2j})$$
$$\|$$
$$-b_1c_{12}^2d_{ijk}-b_1c_{1i}(-b_2d_{1jk}-c_{2k}d_{12j}+c_{2j}d_{12k})-b_1c_{1k}(-b_2d_{1ij}-c_{2j}d_{12i}+c_{2i}d_{12j})+$$
$$+b_1c_{1j}(-b_2d_{1ik}-c_{2k}d_{12i}+c_{2i}d_{12k})$$
$$\|$$
$$-b_1c_{12}^2d_{ijk}-b_1c_{1i}(\tilde{x}\pi g(2,1,j,k)-c_{12}d_{2jk})-b_1c_{1k}(\tilde{x}\pi g(2,1,i,j)-c_{12}d_{2ij})+$$
$$+b_1c_{1j}(\tilde{x}\pi g(2,1,i,k)-c_{12}d_{2ik})$$
$$\|$$
$$c_{12}[c_{12}(-b_1d_{ijk})-c_{1i}(-b_1d_{2jk})-c_{1k}(-b_1d_{2ij})+c_{1j}(-b_1d_{2ik})]$$
$$\|$$
$$c_{12}[c_{12}(\tilde{x}\pi g(1,i,j,k)-c_{1i}d_{1jk}+c_{1j}d_{1ik}-c_{1k}d_{1ij})+$$
$$-c_{1i}(\tilde{x}\pi g(1,2,j,k)-c_{12}d_{1jk}+c_{1j}d_{12k}-c_{1k}d_{12j})+$$
$$-c_{1k}(\tilde{x}\pi g(1,2,i,j)-c_{12}d_{1ij}+c_{1i}d_{12j}-c_{1j}d_{12i})+$$
$$+c_{1j}(\tilde{x}\pi g(1,2,i,k)-c_{12}d_{1ik}+c_{1i}d_{12k}-c_{1k}d_{12i})]$$
$$\|$$
$$0$$

$$\Rightarrow \mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j = d_{ijk}$$

<u>remark</u>: If $n = 3$: $K = (\pi f(1,2,3)) = J$. Theorem 2.3 is also true in this case, for in the proof one sees that

$i(A_1,A_2,A_3) = (a_1,a_2,a_3,b_1,b_2,b_3,c_{12},c_{13},c_{23},d_{123})$ iff

$(a_1,a_2,a_3,b_1,b_2,b_3,c_{12},c_{13},c_{23},d_{123})$ is a zero of $\pi f(1,2,3)$.

<u>lemma 2.7</u>. $\mu_\sigma J \subset J$ for all $\sigma \in S_n$.

<u>proof</u>: $$\mu_\sigma\{\sum_{i_1,i_2,i_3} \alpha^1_{i_1,i_2,i_3} \pi f(i_1,i_2,i_3)+\dots$$
$$\dots+\sum_{i_1,\dots,i_6} \alpha^6_{i_1,\dots,i_6} \pi m(i_1,i_2,i_3,i_4,i_5,i_6)\}=$$
$$= \sum_{i_1,i_2,i_3} \mu_\sigma(\alpha^1_{i_1,i_2,i_3})\pi\mu'_\sigma f(i_1,i_2,i_3)+\dots$$
$$\dots+\sum_{i_1,\dots,i_6} \mu_\sigma(\alpha^6_{i_1,\dots,i_6})\pi\mu'_\sigma m(i_1,\dots,i_6)=$$
$$= \sum_{i_1,i_2,i_3} \mu_\sigma(\alpha^1_{i_1,i_2,i_3})\pi f(\sigma(i_1),\sigma(i_2),\sigma(i_3))+\dots$$
$$\dots+\sum_{i_1,\dots,i_6} \mu_\sigma(\alpha^6_{i_1,\dots,i_6})\pi m(\sigma(i_1),\dots,\sigma(i_6))$$

<u>lemma 2.8</u>. The image of $\pi f(i_1,i_2,i_3)$ and $\pi g(i_1,i_2,i_3,i_4)$ under $\delta : \Sigma \to \Sigma$:

<u>2.8.1</u>. For all $\tau \in S_3$: $\pi f(i_1,i_2,i_3) = \pi f(i_{\tau(1)},i_{\tau(2)},i_{\tau(3)})$.

$\delta\pi f(i,j,k) = \pi f(i,j,k)$ if $1 < i < j < k \leq n$ or $i = 1$, $j = 2$ and $3 \leq k \leq n$.

$\delta\pi f(1,j,k) = \pi f(1,j,k)+\pi f(2,j,k)+2\pi h(1,j,k,2)$ if $3 \leq j < k \leq n$.

<u>2.8.2</u>. For all $\tau \in S_3$: $\pi g(\alpha,i_1,i_2,i_3) = \pm\pi g(\alpha,i_{\tau(1)},i_{\tau(2)},i_{\tau(3)})$.

If $\{i,j,k,l\} \cap \{1,2\} = \emptyset$ then $\delta\pi g(i,j,k,l) = \pi g(i,j,k,l)$

The other cases, where $1 \leq i < j < k < l \leq n$, are summed up in the following schemes:

	$\delta\pi g(i,j,k,l)$
$i=1,\ j=2$	$-\pi g(1,2,k,l)+\pi g(2,1,k,l)$
$i=1,\ j\geq 3$	$\pi\{g(1,j,k,l)+g(2,j,k,l)-l(1,2,j,k,l)-l(2,1,j,k,l)\}$
$i=2$	$\pi g(2,j,k,l)$

	$\delta\pi g(j,i,k,l)$	$\delta\pi g(k,i,j,l)$	$\delta\pi g(l,i,j,k)$
$i=1, j=2$	$\pi g(2,1,k,l)$	$-\pi g(k,1,2,l)$	$-\pi g(l,1,2,k)$
$i=1, j\geq 3$	$\pi g(j,1,k,l)+\pi g(j,2,k,l)$	$\pi g(k,1,j,l)+\pi g(k,2,j,l)$	$\pi g(l,1,j,k)+\pi g(l,2,j,k)$
$i=2$	$-\pi g(j,2,k,l)$	$-\pi g(k,2,j,l)$	$-\pi g(l,2,j,k)$

<u>proof</u>:

(1) $f(i_1,i_2,i_3) = 4W^2_{i_1i_2i_3}-Y_{i_1}Z^2_{i_2i_3}-Y_{i_2}Z^2_{i_1i_3}-Y_{i_3}Z^2_{i_1i_2}+2Z_{i_1i_2}Z_{i_1i_3}Z_{i_2i_3}+$ $+Y_{i_1}Y_{i_2}Y_{i_3}$.

It is sufficient to show: $\pi f(i_1,i_2,i_3) = \pi f(i_{\tau(1)},i_{\tau(2)},i_{\tau(3)})$ if $\tau \in \{(12),(13)\}$.

$$\tau=(12) \Rightarrow \pi f(i_2,i_1,i_3) = \pi(4W^2_{i_2i_1i_3}-Y_{i_2}Z^2_{i_1i_3}-Y_{i_1}Z^2_{i_2i_3}-Y_{i_3}Z^2_{i_2i_1}+$$
$$+2Z_{i_2i_1}Z_{i_2i_3}Z_{i_1i_3}+Y_{i_2}Y_{i_1}Y_{i_3}) =$$
$$= \pi f(i_1,i_2,i_3)$$

$$\tau=(13) \Rightarrow \pi f(i_3,i_2,i_1) = \pi(4W^2_{i_3i_2i_1}-Y_{i_3}Z^2_{i_2i_1}-Y_{i_2}Z^2_{i_3i_1}-Y_{i_1}Z^2_{i_3i_2}+$$
$$+2Z_{i_3i_2}Z_{i_3i_1}Z_{i_2i_1}+Y_{i_3}Y_{i_2}Y_{i_1}) =$$
$$= \pi f(i_1,i_2,i_3)$$

Now let $1 \leq i < j < k \leq n$. Using the definition of δ (cf. 1.5.2), it is evident that $\delta\pi f(i,j,k) = \pi f(i,j,k)$ if $i \geq 3$.

$$\delta\pi f(2,j,k) = \delta(4W^2_{2jk}-Y_2Z^2_{jk}-Y_jZ^2_{2k}-Y_kZ^2_{2j}+2Z_{2j}Z_{2k}Z_{jk}+Y_2Y_jY_k) = \pi f(2,j,k).$$

$$\delta\pi f(1,2,k) = \delta(4W^2_{12k}-Y_1Z^2_{2k}-Y_2Z^2_{1k}-Y_kZ^2_{12}+2Z_{12}Z_{1k}Z_{2k}+Y_1Y_2Y_k) =$$
$$= 4W^2_{12k}-(Y_1+Y_2+2Z_{12})Z^2_{2k}-Y_2(Z_{1k}+Z_{2k})^2-Y_k(Z_{12}+Y_2)^2+$$
$$+2(Z_{12}+Y_2)(Z_{1k}+Z_{2k})Z_{2k}+(Y_1+Y_2+2Z_{12})Y_2Y_k =$$
$$= \pi f(1,2,k).$$

And finally, if $j \geq 3$ then:

$$\begin{aligned}\delta\pi f(1,j,k) &= \delta(4W_{1jk}^2-Y_1Z_{jk}^2-Y_jZ_{1k}^2-Y_kZ_{1j}^2+2Z_{1j}Z_{1k}Z_{jk}+Y_1Y_jY_k) = \\ &= 4(W_{1jk}+W_{2jk})^2-(Y_1+Y_2+2Z_{12})Z_{jk}^2-Y_j(Z_{1k}+Z_{2k})^2-Y_k(Z_{1j}+Z_{2j})^2+ \\ &\qquad +2(Z_{1j}+Z_{2j})(Z_{1k}+Z_{2k})Z_{jk}+(Y_1+Y_2+2Z_{12})Y_jY_k = \\ &= 4W_{1jk}^2-Y_1Z_{jk}^2-Y_jZ_{1k}^2-Y_kZ_{1j}^2+2Z_{1j}Z_{1k}Z_{jk}+Y_1Y_jY_k+ \\ &\qquad +4W_{2jk}^2-Y_2Z_{jk}^2-Y_jZ_{2k}^2-Y_kZ_{2j}^2+2Z_{2j}Z_{2k}Z_{jk}+Y_2Y_jY_k+ \\ &\quad +8W_{1jk}W_{2jk}-2Z_{12}(Z_{jk}^2-Y_jY_k)+2Z_{1k}(Z_{2j}Z_{jk}-Y_jZ_{2k})+ \\ &\quad +2Z_{1j}(Z_{2k}Z_{jk}-Y_kZ_{2j}) = \\ &= \pi f(1,j,k)+\pi f(2,j,k)+2\pi h(1,j,k,2).\end{aligned}$$

(2) $g(\alpha,i_1,i_2,i_3) = -Y_\alpha W_{i_1i_2i_3}+Z_{\alpha i_1}W_{\alpha i_2i_3}-Z_{\alpha i_2}W_{\alpha i_1i_3}+Z_{\alpha i_3}W_{\alpha i_1i_2}.$

$$\begin{aligned}\pi g(\alpha,i_2,i_1,i_3) &= \pi(-Y_\alpha W_{i_2i_1i_3}+Z_{\alpha i_2}W_{\alpha i_1i_3}-Z_{\alpha i_1}W_{\alpha i_2i_3}+Z_{\alpha i_3}W_{\alpha i_2i_1}) = \\ &= -\pi g(\alpha,i_1,i_2,i_3).\\ \pi g(\alpha,i_3,i_2,i_1) &= \pi(-Y_\alpha W_{i_3i_2i_1}+Z_{\alpha i_3}W_{\alpha i_2i_1}-Z_{\alpha i_2}W_{\alpha i_3i_1}+Z_{\alpha i_1}W_{\alpha i_3i_2}) = \\ &= -\pi g(\alpha,i_1,i_2,i_3).\end{aligned}$$

From these identities it follows that

$\pi g(\alpha,i_{\tau(1)},i_{\tau(2)},i_{\tau(3)}) = \pm\pi g(\alpha,i_1,i_2,i_3)$ for all $\tau \in S_3$.

In view of the definition of δ it is evident that

$\delta\pi g(i,j,k,l) = \pi g(i,j,k,l)$ if $\{i,j,k,l\} \cap \{1,2\} = \emptyset$. Now let

$1 \le i < j < k < l \le n$. Distinguish three cases:

(i) $2 = i < j < k < l$. Then:

$$\begin{aligned}\delta\pi g(2,j,k,l) &= \pi\delta'(-Y_2W_{jkl}+Z_{2j}W_{2kl}-Z_{2k}W_{2jl}+Z_{2l}W_{2jk}) = \pi g(2,j,k,l)\\ \delta\pi g(j,2,k,l) &= \pi\delta'(-Y_jW_{2kl}+Z_{j2}W_{jkl}-Z_{jk}W_{j2l}+Z_{jl}W_{j2k}) = -\pi g(j,2,k,l)\\ \delta\pi g(k,2,j,l) &= \pi\delta'(-Y_kW_{2jl}+Z_{k2}W_{kjl}-Z_{kj}W_{k2l}+Z_{kl}W_{k2j}) = -\pi g(k,2,j,l)\\ \delta\pi g(l,2,j,k) &= \pi\delta'(-Y_lW_{2jk}+Z_{l2}W_{ljk}-Z_{lj}W_{l2k}+Z_{lk}W_{l2j}) = -\pi g(l,2,j,k)\end{aligned}$$

(ii) $i = 1,\ 2 = j < k < l$. Then:

$$
\begin{aligned}
\delta\pi g(1,2,k,l) &= \delta(-Y_1W_{2kl}+Z_{12}W_{1kl}-Z_{1k}W_{12l}+Z_{1l}W_{12k}) \\
&= (Y_1+Y_2+2Z_{12})W_{2kl}-(Z_{12}+Y_2)(W_{1kl}+W_{2kl})+(Z_{1k}+Z_{2k})W_{12l}+ \\
&\qquad -(Z_{1l}+Z_{2l})W_{12k} \\
&= Y_1W_{2kl}-Z_{12}W_{1kl}+Z_{1k}W_{12l}-Z_{1l}W_{12k}-Y_2W_{1kl}+Z_{12}W_{2kl}-Z_{2l}W_{12k}+ \\
&\qquad +Z_{2k}W_{12l} \\
&= -\pi g(1,2,k,l)+\pi g(2,1,k,l).
\end{aligned}
$$

$$
\begin{aligned}
\delta\pi(2,1,k,l) &= \delta\{\delta\pi g(1,2,k,l)+\pi g(1,2,k,l)\} \\
&= \pi g(1,2,k,l)+\delta\pi g(1,2,k,l) \qquad (\delta^2 = 1) \\
&= \pi g(2,1,k,l).
\end{aligned}
$$

$$
\begin{aligned}
\delta\pi g(k,1,2,l) &= \pi\delta'(-Y_kW_{12l}+Z_{k1}W_{k2l}-Z_{k2}W_{k1l}+Z_{kl}W_{k12}) \\
&= \pi\{Y_kW_{12l}-(Z_{k1}+Z_{k2})W_{k2l}+Z_{k2}(W_{k1l}+W_{k2l})-Z_{kl}W_{k12}\} \\
&= -\pi g(k,1,2,l).
\end{aligned}
$$

$$
\begin{aligned}
\delta\pi g(l,1,2,k) &= \pi\delta'(-Y_lW_{12k}+Z_{l1}W_{l2k}-Z_{l2}W_{l1k}+Z_{lk}W_{l12}) \\
&= \pi\{Y_lW_{12k}-(Z_{l1}+Z_{l2})W_{l2k}+Z_{l2}(W_{l1k}+W_{l2k})-Z_{lk}W_{l12}\} \\
&= -\pi g(l,1,2,k).
\end{aligned}
$$

(iii) $i = 1,\ 3 \leq j < k < l$. Then:

$$
\begin{aligned}
\delta\pi g(1,j,k,l) &= \delta(-Y_1W_{jkl}+Z_{1j}W_{1kl}-Z_{1k}W_{1jl}+Z_{1l}W_{1jk}) \\
&= -(Y_1+Y_2+2Z_{12})W_{jkl}+(Z_{1j}+Z_{2j})(W_{1kl}+W_{2kl})-(Z_{1k}+Z_{2k})(W_{1jl}+W_{2jl})+ \\
&\qquad +(Z_{1l}+Z_{2l})(W_{1jk}+W_{2jk}) \\
&= -Y_1W_{jkl}+Z_{1j}W_{1kl}-Z_{1k}W_{1jl}+Z_{1l}W_{1jk}+ \\
&\qquad -Y_2W_{jkl}+Z_{2j}W_{2kl}-Z_{2k}W_{2jl}+Z_{2l}W_{2jk}+ \\
&\qquad -2Z_{12}W_{jkl}+Z_{1j}W_{2kl}+Z_{2j}W_{1kl}-Z_{1k}W_{2jl}-Z_{2k}W_{1jl}+Z_{1l}W_{2jk}+ \\
&\qquad +Z_{2l}W_{1jk} \\
&= \pi g(1,j,k,l)+\pi g(2,j,k,l)-\pi l(1,2,j,k,l)-\pi l(2,1,j,k,l).
\end{aligned}
$$

$$\delta\pi g(j,1,k,l) = \pi\delta'(-Y_jW_{1kl}+Z_{j1}W_{jkl}-Z_{jk}W_{j1l}+Z_{jl}W_{j1k})$$
$$= \pi\{-Y_j(W_{1kl}+W_{2kl})+(Z_{j1}+Z_{j2})W_{jkl}-Z_{jk}(W_{j1l}+W_{j2l})+$$
$$+Z_{jl}(W_{j1k}+W_{j2k})\}$$
$$= \pi g(j,1,k,l)+\pi g(j,2,k,l).$$

Analogously: $\delta\pi g(k,1,j,l) = \pi g(k,1,j,l)+\pi g(k,2,j,l)$

$\delta\pi g(l,1,j,k) = \pi g(l,1,j,k)+\pi g(l,2,j,k)$.

lemma 2.9. $\delta K+\delta\mu_{(132)}K+\delta\mu_{(23)}K+\delta\mu_{(132)}\delta K \subset J$.

proof:

(1) $\delta K \subset J$? Because of lemma 2.8, we only have to show that $\delta\pi h(i,1,2,j) \in J$ if $3 \le i < j \le n$.

$$\delta\pi h(i,1,2,j) = \delta\{4W_{12i}W_{12j}-Z_{ij}(Z_{12}^2-Y_1Y_2)+Z_{1j}(Z_{12}Z_{2i}-Y_2Z_{1i})+$$
$$+Z_{2j}(Z_{12}Z_{1i}-Y_1Z_{2i})\}$$
$$= 4W_{12i}W_{12j}-Z_{ij}\{(Z_{12}+Y_2)^2-(Y_1+Y_2+2Z_{12})Y_2\}+$$
$$+(Z_{1j}+Z_{2j})\{(Z_{12}+Y_2)Z_{2i}-Y_2(Z_{1i}+Z_{2i})\}+$$
$$-Z_{2j}\{-(Z_{12}+Y_2)(Z_{1i}+Z_{2i})+(Y_1+Y_2+2Z_{12})Z_{2i})\}$$
$$= \pi h(i,1,2,j) \in J.$$

(2) $\delta\mu_{(132)}K \subset J$?

In general we have for all $\sigma \in S_n$:

$$\delta\mu_\sigma\pi f(i,j,k) = \delta\pi\mu_\sigma' f(i,j,k) = \delta\pi f(\sigma(i),\sigma(j),\sigma(k)) \in J \quad \text{(cf. 2.8.1)}$$
$$\delta\mu_\sigma\pi g(i,j,k,l) = \delta\pi g(\sigma(i),\sigma(j),\sigma(k),\sigma(l)) \in J \quad \text{(cf. 2.8.2)}$$

So it remains to show that $\delta\mu_{(132)}\pi h(i,1,2,j) \in J$ if $3 \le i < j \le n$.

Distinguish two cases:

(2.1) If $i = 3$ then

$$\delta\mu_{(132)}\pi h(3,1,2,j) = \delta\pi\mu'_{(132)}h(3,1,2,j) = \delta\pi h(2,3,1,j) =$$

$$= \delta\{-4W_{123}W_{13j}-Z_{2j}(Z_{13}^2-Y_1Y_3)+Z_{3j}(Z_{13}Z_{12}-Y_1Z_{23})+Z_{1j}(Z_{13}Z_{23}-Y_3Z_{12})\}$$

$$= 4W_{123}(W_{13j}+W_{23j})+Z_{2j}\{(Z_{13}+Z_{23})^2-(Y_1+Y_2+2Z_{12})Y_3\}+$$
$$+Z_{3j}\{-(Z_{13}+Z_{23})(Z_{12}+Y_2)+(Y_1+Y_2+2Z_{12})Z_{23}\}+$$
$$+(Z_{1j}+Z_{2j})\{-(Z_{13}+Z_{23})Z_{23}+Y_3(Z_{12}+Y_2)\}$$

$$= -\pi h(2,3,1,j)+\pi h(1,2,3,j) \in J.$$

(2.2) If $3 < i < j \leq n$ then

$$\delta\mu_{(132)}\pi h(i,1,2,j) = \delta\pi h(i,3,1,j) =$$

$$= \delta\{4W_{13i}W_{13j}-Z_{ij}(Z_{13}^2-Y_1Y_3)+Z_{3j}(Z_{13}Z_{1i}-Y_1Z_{3i})+Z_{1j}(Z_{13}Z_{3i}-Y_3Z_{1i})\}$$

$$= 4(W_{13i}+W_{23i})(W_{13j}+W_{23j})-Z_{ij}\{(Z_{13}+Z_{23})^2-(Y_1+Y_2+2Z_{12})Y_3\}+$$
$$+Z_{3j}\{(Z_{13}+Z_{23})(Z_{1i}+Z_{2i})-(Y_1+Y_2+2Z_{12})Z_{3i}\}+$$
$$+(Z_{1j}+Z_{2j})\{(Z_{13}+Z_{23})Z_{3i}-Y_3(Z_{1i}+Z_{2i})\}$$

$$= 4W_{13i}W_{13j}-Z_{ij}(Z_{13}^2-Y_1Y_3)+Z_{3j}(Z_{13}Z_{1i}-Y_1Z_{3i})+Z_{1j}(Z_{13}Z_{3i}-Y_3Z_{1i})+$$
$$+4W_{23i}W_{23j}-Z_{ij}(Z_{23}^2-Y_2Y_3)+Z_{3j}(Z_{23}Z_{2i}-Y_2Z_{3i})+Z_{2j}(Z_{23}Z_{3i}-Y_3Z_{2i})+$$
$$+4W_{13i}W_{23j}+Y_3(Z_{12}Z_{ij}-Z_{1j}Z_{2i})+Z_{13}(Z_{3j}Z_{2i}-Z_{23}Z_{ij})+Z_{3i}(Z_{23}Z_{1j}-Z_{12}Z_{3j})+$$
$$+4W_{13j}W_{23i}+Y_3(Z_{12}Z_{ij}-Z_{1i}Z_{2j})+Z_{13}(Z_{3i}Z_{2j}-Z_{23}Z_{ij})+Z_{3j}(Z_{23}Z_{1i}-Z_{12}Z_{3i})$$

$$= \pi h(i,3,1,j)+\pi h(i,3,2,j)+\pi k(3,i,1,j,2)+\pi k(3,j,1,i,2) \in J.$$

(3) $\delta\mu_{(23)}K \subset J$?

Again we only have to prove that $\delta\mu_{(23)}\pi h(i,1,2,j) \in J$ for all $3 \leq i < j \leq n$.

First notice that $\pi h(i_1,i_2,i_3,i_4) = \pi h(i_1,i_3,i_2,i_4) = \pi h(i_4,i_2,i_3,i_1)$.

Therefore:

(3.1) If $i \neq 3$: $\delta\mu_{(23)}\pi h(i,1,2,j) = \delta\pi h(i,1,3,j) = \delta\pi h(i,3,1,j) \in J$ (cf. (2.2) above).

(3.2) If $i = 3$: $\delta\mu_{(23)}\pi h(3,1,2,j) = \delta\pi h(2,1,3,j) = \delta\pi h(2,3,1,j) \in J$ (cf. (2.1) above).

(4) $\delta\mu_{(132)}\delta K \subset J$?

(4.1) If $k \geq 3$ then $\delta\mu_{(132)}\delta\pi f(1,2,k) = \delta\mu_{(132)}\pi f(1,2,k) \in J$ (cf. 2.8.1 and (2) above).

(4.2) In view of lemma 2.8.2 and (2) above it is evident that $\delta\mu_{(132)}\delta\pi g(i,j,k,l) \in J$ if $\delta\mu_{(132)}\pi\{l(1,2,i,j,k)+l(2,1,i,j,k)\} \in J$ for all $3 \leq i < j < k \leq n$.

$$\pi l(1,2,i,j,k)+\pi l(2,1,i,j,k) = 2Z_{12}W_{ijk}-Z_{1i}W_{2jk}-Z_{2i}W_{1jk}+Z_{1j}W_{2ik}+Z_{2j}W_{1ik}+ -Z_{1k}W_{2ij}-Z_{2k}W_{1ij}.$$

Distinguish two cases:

(a) $3 < i < j < k \leq n$

$$\begin{aligned}
\delta\mu_{(132)}\pi\{l(1,2,i,j,k)+l(2,1,i,j,k)\} &= \\
&= \delta(2Z_{13}W_{ijk}-Z_{3i}W_{1jk}-Z_{1i}W_{3jk}+Z_{3j}W_{1ik}+Z_{1j}W_{3ik}-Z_{3k}W_{1ij}-Z_{1k}W_{3ij}) \\
&= 2(Z_{13}+Z_{23})W_{ijk}-Z_{3i}(W_{1jk}+W_{2jk})-(Z_{1i}+Z_{2i})W_{3jk}+ \\
&\quad +Z_{3j}(W_{1ik}+W_{2ik})+(Z_{1j}+Z_{2j})W_{3ik}-Z_{3k}(W_{1ij}+W_{2ij})-(Z_{1k}+Z_{2k})W_{3ij} \\
&= 2Z_{13}W_{ijk}-Z_{3i}W_{1jk}-Z_{1i}W_{3jk}+Z_{3j}W_{1ik}+Z_{1j}W_{3ik}-Z_{3k}W_{1ij}-Z_{1k}W_{3ij}+ \\
&\quad +2Z_{23}W_{ijk}-Z_{3i}W_{2jk}-Z_{2i}W_{3jk}+Z_{3j}W_{2ik}+Z_{2j}W_{3ik}-Z_{3k}W_{2ij}-Z_{2k}W_{3ij} \\
&= \pi l(3,1,i,j,k)+\pi l(1,3,i,j,k)+\pi l(3,2,i,j,k)+\pi l(2,3,i,j,k) \in J.
\end{aligned}$$

(b) $3 = i < j < k \leq n$

$$\begin{aligned}
\delta\mu_{(132)}\pi\{l(1,2,3,j,k)+l(2,1,3,j,k)\} &= \\
&= \delta(2Z_{13}W_{2jk}-Z_{23}W_{1jk}-Z_{12}W_{3jk}+Z_{3j}W_{12k}-Z_{1j}W_{23k}-Z_{3k}W_{12j}+Z_{1k}W_{23j}) \\
&= -2(Z_{13}+Z_{23})W_{2jk}+Z_{23}(W_{1jk}+W_{2jk})+(Z_{12}+Y_{2})W_{3jk}-Z_{3j}W_{12k}+ \\
&\quad +(Z_{1j}+Z_{2j})W_{23k}+Z_{3k}W_{12j}-(Z_{1k}+Z_{2k})W_{23j} \\
&= -2Z_{13}W_{2jk}+Z_{23}W_{1jk}+Z_{12}W_{3jk}-Z_{3j}W_{12k}+Z_{1j}W_{23k}+Z_{3k}W_{12j}-Z_{1k}W_{23j}+ \\
&\quad +Y_{2}W_{3jk}-Z_{23}W_{2jk}+Z_{2j}W_{23k}-Z_{2k}W_{23j} \\
&= -\pi l(3,1,2,j,k)-\pi l(1,3,2,j,k)-\pi g(2,3,j,k) \in J.
\end{aligned}$$

(4.3) For all $3 \leq i < j \leq n$ we have:

$\delta\mu_{(132)}\delta\pi h(i,1,2,j) \underset{(1)}{=} \delta\mu_{(132)}\pi h(i,1,2,j) \underset{(2)}{\in} J.$

remark: With some perseverance it can be proved that $\delta J \subset J$, a consequence of which lemma 2.9 would be (cf. 2.17 below).

proposition 2.10. $J_0 = K+\mu_{(12)}K+\delta K \subset J$. If $x \in V(J_0)-V(Z_{12}^2-Y_1Y_2)$ then there exists $A \in M(2,k)^n$ such that $i(A) = x$.

proof: $x \notin V(Z_{12}^2-Y_1Y_2)$ iff $x \notin V(Y_1)$ or $x \notin V(Y_2)$ or $x \notin V(Y_1+Y_2+2Z_{12})$. Therefore we distinguish three possibilities.

(1) If $x \notin V(Y_1)$ then $x \in V(K)-V(Z_{12}^2-Y_1Y_2) \cup V(Y_1)$. In virtue of th.2.6. there exists $A \in M(2,k)^n$ such that $i(A) = x$.

(2) If $x \notin V(Y_2)$ we define $y = \overline{\mu}_{(12)}(x)$. I claim that $y \in V(K)-V(Z_{12}^2-Y_1Y_2) \cup V(Y_1)$.

This can be seen as follows:

$$\begin{cases} t \in K \Rightarrow \tilde{y}(t) = \tilde{x}\mu_{(12)}t = 0 \text{ (cf. 1.3 and the definition of } J_0 \text{ up here)} \\ \tilde{y}(Z_{12}^2-Y_1Y_2) = \tilde{x}\mu_{(12)}(Z_{12}^2-Y_1Y_2) = \tilde{x}(Z_{12}^2-Y_1Y_2) \neq 0 \\ \tilde{y}(Y_1) = \tilde{x}\mu_{(12)}(Y_1) = \tilde{x}(Y_2) \neq 0 \end{cases}$$

So from th.2.6. it follows that $y = i(B)$ for some $B \in M(2,k)^n$.

Put $A = \overline{\nu}_{(12)}B$ then $i(A) = i\overline{\nu}_{(12)}(B) = \overline{\mu}_{(12)}i(B) = \overline{\mu}_{(12)}y = \overline{\mu}_{(12)}\overline{\mu}_{(12)}x = x$ (cf. 1.4.)

(3) If $x \notin V(Y_1+Y_2+2Z_{12})$ we define $y = \overline{\delta}(x)$. Then $y \in V(K)-V(Z_{12}^2-Y_1Y_2) \cup V(Y_1)$ because:

$$\begin{cases} t \in K \Rightarrow \delta t \in J_0 \Rightarrow \tilde{y}(t) = \tilde{x}\delta(t) = 0 \quad \text{(cf. 1.6.)} \\ \tilde{y}(Z_{12}^2-Y_1Y_2) = \tilde{x}\delta(Z_{12}^2-Y_1Y_2) = \tilde{x}\{(Z_{12}+Y_2)^2-(Y_1+Y_2+2Z_{12})Y_2\} = \\ \qquad\qquad = \tilde{x}(Z_{12}^2-Y_1Y_2) \neq 0 \\ \tilde{y}(Y_1) = \tilde{x}\delta(Y_1) = \tilde{x}(Y_1+Y_2+2Z_{12}) \neq 0. \end{cases}$$

Consequently $y = i(B)$ for some $B \in M(2,k)^n$ (2.6). Putting $A = \overline{d}(B)$ we have $i(A) = i\overline{d}(B) = \overline{\delta}i(B) = \overline{\delta}(y) = \overline{\delta}\overline{\delta}(x) = x$ (cf. 1.7.)

<u>lemma 2.11</u>. $V(\Delta f_{123}) = V(f_{123}, Z_{12}^2-Y_1Y_2, Z_{13}^2-Y_1Y_3, Z_{23}^2-Y_2Y_3,$
$Z_{12}^2-Y_1Y_2+Z_{13}^2-Y_1Y_3+2(Z_{12}Z_{13}-Y_1Z_{23}))$.

<u>proof</u>:

$V(\Delta f_{123}) = V(W_{123}, f_{123}, Z_{12}^2-Y_1Y_2, Z_{13}^2-Y_1Y_3, Z_{23}^2-Y_2Y_3, Z_{12}Z_{13}-Y_1Z_{23})$ and
$f_{123} = 4W_{123}^2-Y_3(Z_{12}^2-Y_1Y_2)-Y_2(Z_{13}^2-Y_1Y_3)+Y_1(Z_{23}^2-Y_2Y_3)+2Z_{23}(Z_{12}Z_{13}-Y_1Z_{23})$
(cf. III, §2).

<u>proposition 2.12</u>. If $x \in V(J)-V(\Delta f_{123})$ there exists $A \in M(2,k)^n$ such that $i(A) = x$.

<u>proof</u>: Applying the above lemma we distinguish four cases.

Case 1: $x \in V(J)-V(Z_{12}^2-Y_1Y_2)$. Then the assertion of the proposition follows from 2.10.

Case 2: $x \in V(J)-V(Z_{13}^2-Y_1Y_3)$. Then $y = \overline{\mu}_{(23)}(x) \in V(J)-V(Z_{12}^2-Y_1Y_2)$
because: (a) $t \in J \Rightarrow \mu_{(23)} \in J \Rightarrow \tilde{y}(t) = \tilde{x}\mu_{(23)}(t) = 0$ (cf.2.7 and 1.3.)
(b) $\tilde{y}(Z_{12}^2-Y_1Y_2) = \tilde{x}\mu_{(23)}(Z_{12}^2-Y_1Y_2) = \tilde{x}(Z_{13}^2-Y_1Y_3) \neq 0$

So $y = i(B)$ for some $B \in M(2,k)^n$ (2.10). Put $A = \overline{\nu}_{(23)}(B)$ then $i(A) = x$.

Case 3: $x \in V(J)-V(Z_{23}^2-Y_2Y_3)$. Analogous to case 2 taking $y = \overline{\mu}_{(13)}(x)$.

Case 4: $x \in V(J)-V(Z_{12}^2-Y_1Y_2+Z_{13}^2-Y_1Y_3+2(Z_{12}Z_{13}-Y_1Z_{23}))$. I claim that $y = \overline{\mu}_{(123)}\overline{\delta}\overline{\mu}_{(132)}(x)$ belongs to $V(J_0)-V(Z_{12}^2-Y_1Y_2)$ where
$J_0 = K+\mu_{(12)}K+\delta K$

This follows from:

$$(a)\begin{cases} t \in K \Rightarrow \delta\mu_{(132)}(t) \in J \text{ (2.9)} \Rightarrow \mu_{(123)}\delta\mu_{(132)}(t) \in J \text{ (2.7)}. \\ t = \mu_{(12)}s \in \mu_{(12)}K \Rightarrow \delta\mu_{(132)}(t) = \delta\mu_{(23)}s \in J \Rightarrow \mu_{(123)}\delta\mu_{(132)}(t) \in J. \\ t = \delta s \in \delta K \Rightarrow \delta\mu_{(132)}(t) = \delta\mu_{(132)}\delta s \in J \text{ (2.9.)} \Rightarrow \mu_{(123)}\delta\mu_{(132)}(t) \in J. \end{cases}$$

Therefore $\mu_{(123)}\delta\mu_{(132)}(t) \in J$ if $t \in J_0$.

(b) $y \in V(J_0)$ because $\tilde{y}(t) = \tilde{x}[\mu_{(123)}\delta\mu_{(132)}(t)] \underset{(a)}{=} 0$ for all $t \in J_0$ (cf. 1.3. and 1.6.)

(c) $$\begin{aligned}\tilde{y}(Z_{12}^2-Y_1Y_2) &= \tilde{x}\mu_{(123)}\delta\mu_{(132)}(Z_{12}^2-Y_1Y_2)\\ &= \tilde{x}\mu_{(123)}\delta(Z_{13}^2-Y_1Y_3)\\ &= \tilde{x}\mu_{(123)}[(Z_{13}+Z_{23})^2-(Y_1+Y_2+2Z_{12})Y_3]\\ &= \tilde{x}[(Z_{12}+Z_{13})^2-(Y_2+Y_3+2Z_{23})Y_1]\\ &= \tilde{x}[Z_{12}^2-Y_1Y_2+Z_{13}^2-Y_1Y_3+2(Z_{12}Z_{13}-Y_1Z_{23})] \neq 0.\end{aligned}$$

According to prop. 2.10 there exists $B = (B_1,\dots,B_n)$ such that $y = i(B)$. Let $A = \overline{\nu}_{(123)}\overline{d}\overline{\nu}_{(132)}(B) = (B_1,B_2+B_3,-B_3,B_4,\dots,B_n)$ then we have
$i(A) = i\overline{\nu}_{(123)}\overline{d}\overline{\nu}_{(132)}B = \overline{\mu}_{(123)}\overline{\delta}\overline{\mu}_{(132)}i(B) =$
$= \overline{\mu}_{(123)}\overline{\delta}\overline{\mu}_{(132)}\overline{\mu}_{(123)}\overline{\delta}\overline{\mu}_{(132)}(x) = x$ (cf. 1.4. and 1.7.)

<u>theorem 2.13</u>. $x \in V(J)$ and $x \notin V(\Delta f_{ijk})$ for some $1 \leq i < j < k \leq n$ if and only if there exists an irreducible element A in $M(2,k)^n$ such that $i(A) = x$.

<u>proof</u>:

(1) Assume $x \in V(J)-V(\Delta f_{ijk})$. Define $\sigma \in S_n$ by $\sigma = (3k)(2j)(1i)$ then $\sigma^{-1}(1) = i$, $\sigma^{-1}(2) = j$ and $\sigma^{-1}(3) = k$ whatever i,j and k may be. Put $y = \overline{\mu}_\sigma(x)$ then $y \in V(J)-V(\Delta f_{123})$ because:

(a) If $t \in J$ then $\mu_{\sigma^{-1}}(t) \in J$ and $\tilde{y}(t) = \tilde{x}\mu_{\sigma^{-1}}(t) = 0$. So $y \in V(J)$.

(b) $y \in V(\Delta f_{123}) = V(f_{123},W_{123},Z_{12}^2-Y_1Y_2,Z_{13}^2-Y_1Y_3,Z_{23}^2-Y_2Y_3)$ iff $x \in V(f_{ijk},W_{ijk},Z_{ij}^2-Y_iY_j,Z_{ik}^2-Y_iY_k,Z_{jk}^2-Y_jY_k) = V(\Delta f_{ijk})$.

Using prop. 2.12. $y = i(B)$ for some $B \in M(2,k)^n$.

Let $A = \overline{\nu}_{\sigma^{-1}}(B)$ then $i(A) = i\overline{\nu}_{\sigma^{-1}}(B) = \overline{\mu}_{\sigma^{-1}}\overline{\mu}_\sigma(x) = x$. Notice that A is irreducible because $i(A) \notin V(\Delta f_{ijk})$ (cf. II.1.6. and the proof of II.2.3.)

(2) Suppose that $A \in M(2,k)^n$ is irreducible and let $x = i(A)$.

Because $J \subset \ker \alpha$ (1.10), $\tilde{x}(t) = \widetilde{i(A)}(t) = \tilde{A}(\alpha(t)) = 0$ for all $t \in J$ (cf. I.1.10). Therefore $x \in V(J)$. Now A is irreducible so $x = i(A) \notin V(\Delta f_{ijk})$ for some $1 \leq i < j < k \leq n$ (cf. II.2.3).

Recapitulating:

In theorem 2.5. we have shown: if $x \in V(J)$ and $x \in V(\Delta f_{ijk})$ for all $1 \leq i < j < k \leq n$ then there exists A such that $i(A) = x$. Combining this with the theorem up here we obtain: if $x \in V(J)$ then $x = i(A)$ for some $A \in M(2,k)^n$. Applying lemma 2.1. it then follows:

$$\boxed{\ker \alpha = \sqrt{J}}$$

In order to prove $J = \sqrt{J}$ we want to use the following lemma

lemma 2.14. Let $J = (f_1,\ldots,f_N)$ be an ideal in $k[T_1,\ldots,T_m]$ such that $\sqrt{J}$ is a prime ideal. Let p be the dimension of the variety $\operatorname{Spec} {}^{k[T_1,\ldots,T_m]}/_{\sqrt{J}}$. Let J' denote "the Jacobimatrix belonging to J", i.e.,

$$J' = \left(\frac{\delta f_i}{\delta T_j}\right)_{1 \leq i \leq N,\, 1 \leq j \leq m}$$

Suppose there exists a closed point y in $V(J) \subset k^m$ such that $\operatorname{rank} J'(y) = m-p$. Then $J = \sqrt{J}$.

proof:

Suppose $J \neq \sqrt{J}$. Then there exists $g \in k[T_1,\ldots,T_m]$ satisfying $g \notin J$ and $g^2 \in J$. Put $Z = \operatorname{Spec} {}^{k[T_1,\ldots,T_m]}/_{J}$; Z is an irreducible k-scheme but not reduced. y corresponds to a closed point $\bar{y}$ of Z. Since $\operatorname{rank} J'(y) = m-p$, Z is smooth at $\bar{y}$. In particular $\mathcal{O}_{Z,\bar{y}}$ is an integral domain [8], so it certainly contains no nilpotents ($\neq 0$).

Now $\bar{g} = g+J \in {}^{k[T_1,\ldots,T_m]}/_{J}$ induces an element γ in $\mathcal{O}_{Z,\bar{y}}$. $\gamma \neq 0$ because if $\gamma = 0$, $\bar{g}$ vanishes in a neighbourhood of $\bar{y}$; but then $\bar{g}$

vanishes everywhere on Z because of the irreducibility of Z. So $\overline{g} = 0$ or equivalently $g \in J$. Contradiction.

On the other hand $\dot{\gamma}^2 = 0$, because $g^2 \in J$, and therefore γ is a nilpotent element of $\mathcal{O}_{Z,\overline{y}}$. Contradiction.

<u>lemma 2.15</u>. $Y = \operatorname{Spec} R^G \simeq \operatorname{Spec} {}^{\Sigma}/\sqrt{J}$ is an irreducible variety of dimension $4n-3$.

<u>proof</u>: Because $\sqrt{J} = \ker \alpha$ is prime, Y is an irreducible variety. The morphism $\phi : X \to Y$ induced by the inclusion $R^G \subset R$ is a dominant morphism of varieties. Say $r = \dim X - \dim Y = 4n - \dim Y$. There exists a nonempty open subset U in Y such that - for all $y \in U$ - $\phi^{-1}(y)$ is a nonempty "pure" r-dimensional set, i.e., all its components have the dimension r [8].

Now Y_S is a nonempty open subset of Y, Y is irreducible, so $U \cap Y_S \neq \emptyset$, i.e., there exists $y \in Y_S$ such that $\phi^{-1}(y)$ is pure r-dimensional. Since $\phi^{-1}(y)$ is the orbit of a closed point x corresponding to an irreducible element A of $M(2,k)^n$, we get:

$$r = \dim \phi^{-1}(y) = \dim G(x) = \dim G - \dim S(x) = 4-1 = 3 \text{ (cf.II.3.2)}.$$

Hence $\dim Y = 4n-r = 4n-3$.

<u>theorem 2.16</u>. $J = \sqrt{J} = \ker \alpha$.

<u>proof</u>:

Let J' be the matrix (with coefficients in Σ) defined by:

$$J' = \left(\frac{\delta t}{\delta X_1}\cdots\frac{\delta t}{\delta X_n}\ \frac{\delta t}{\delta Y_1}\cdots\frac{\delta t}{\delta Y_n}\ \frac{\delta t}{\delta Z_{12}}\cdots\frac{\delta t}{\delta Z_{n-1,n}}\ \frac{\delta t}{\delta W_{123}}\cdots\frac{\delta t}{\delta W_{n-2,n-1,n}}\right)$$

where t runs through the generators of J given in 1.8. According to lemma 2.14 $J = \sqrt{J}$ if we can find a closed point $y \in V(J)$ such that $\operatorname{rank} J'(y) = m - \dim Y = 2n+\binom{n}{2}+\binom{n}{3}-(4n-3) = \binom{n}{3}+\frac{1}{2}(n-2)(n-3)$.

Let $y = i(\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\0&0\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\ldots,\begin{pmatrix}0&0\\0&0\end{pmatrix}) = (0,\ldots,0,1,0,\ldots,0)$ with $\tilde{y}(Z_{12}) = 1$. We claim that rank $J'(y) = \binom{n}{3}+\frac{1}{2}(n-2)(n-3)$.

Since rank $J'(z) \leq \binom{n}{3}+\frac{1}{2}(n-2)(n-3)$ for all $z \in V(J)$, we only have to show that rank $J'(y) \geq \binom{n}{3}+\frac{1}{2}(n-2)(n-3)$.

(1) Since X_i $(1 \leq i \leq n)$ does not occur in the generators of J given in 1.8, we certainly have $\frac{\delta t}{\delta X_i} = 0$ whenever t is such a generator.

(2) $\frac{\delta}{\delta Y_i}\pi m(i_1,\ldots,i_6) = \frac{\delta}{\delta Y_i}\pi l(i_1,\ldots,i_5) = 0$ for all $1 \leq i \leq n$.

$\frac{\delta}{\delta Y_i}\pi k(i_1,\ldots,i_5) = \delta_{ii_1}\pi(Z_{i_2i_4}Z_{i_3i_5}-Z_{i_2i_5}Z_{i_3i_4})$; only one coordinate of y is different zero, so $\tilde{y}(\frac{\delta}{\delta Y_i}\pi k(i_1,\ldots,i_5)) = 0$ (δ_{ii_1} is the Kronecker-delta).

$\frac{\delta}{\delta Y_i}\pi h(i_1,i_2,i_3,i_4)=\delta_{ii_2}[\pi(Y_{i_3}Z_{i_1i_4}-Z_{i_1i_3}Z_{i_3i_4})]+\delta_{ii_3}[\pi(Y_{i_2}Z_{i_1i_4}-Z_{i_1i_2}Z_{i_2i_4})$
and

$\frac{\delta}{\delta Y_i}\pi g(i_1,i_2,i_3,i_4) \in k[W_{123},\ldots,W_{n-2,n-1,n}]$, so we have

$\tilde{y}(\frac{\delta}{\delta Y_i}\pi h(i_1,i_2,i_3,i_4)) = \tilde{y}(\frac{\delta}{\delta Y_i}\pi g(i_1,i_2,i_3,i_4)) = 0.$

$$\frac{\delta}{\delta Y_i}\pi f(i_1,i_2,i_3)=\delta_{ii_1}[\pi(-Z^2_{i_2i_3}+Y_{i_2}Y_{i_3})]+\delta_{ii_2}[\pi(-Z^2_{i_1i_3}+Y_{i_1}Y_{i_3})]+$$
$$+\delta_{ii_3}[\pi(-Z^2_{i_1i_2}+Y_{i_1}Y_{i_2})].$$

Therefore: $\tilde{y}(\frac{\delta}{\delta Y_i}\pi f(i_1,i_2,i_3)) = 0$ if $\{1,2\} \not\subset \{i_1,i_2,i_3\}$ and $\tilde{y}(\frac{\delta}{\delta Y_i}\pi f(1,2,k)) = -\delta_{ik}$.

(3) Let $1 \leq i < j \leq n$.

$\frac{\delta}{\delta Z_{ij}}\pi f(i_1,i_2,i_3)$, $\frac{\delta}{\delta Z_{ij}}\pi k(i_1,\ldots,i_5)$ and $\frac{\delta}{\delta Z_{ij}}\pi m(i_1,\ldots,i_6)$ only contain "mixed" terms,i.e.,terms of form $Z_{pq}Z_{rs}$ or Y_pZ_{qr}.

Hence $\tilde{y}(\frac{\delta}{\delta Z_{ij}}\pi f(i_1,i_2,i_3)) = \tilde{y}(\frac{\delta}{\delta Z_{ij}}\pi k(i_1,\ldots,i_5)) = \tilde{y}(\frac{\delta}{\delta Z_{ij}}\pi m(i_1,\ldots,i_6) = 0.$

$\frac{\delta}{\delta Z_{ij}}\pi g(i_1,\ldots,i_4)$ and $\frac{\delta}{\delta Z_{ij}}\pi l(i_1,\ldots,i_5)$ are elements of $k[W_{123},\ldots,W_{n-2,n-1,n}]$.

Therefore $\tilde{y}(\frac{\delta}{\delta Z_{ij}}\pi g(i_1,\ldots,i_4)) = \tilde{y}(\frac{\delta}{\delta Z_{ij}}\pi l(i_1,\ldots,i_5)) = 0$.

If $\{i,j\} \neq \{i_1,i_4\}$, $\frac{\delta}{\delta Z_{ij}}\pi h(i_1,i_2,i_3,i_4)$ only contains mixed terms (or equals zero) and if $\{i,j\} = \{i_1,i_4\}$ then $\frac{\delta}{\delta Z_{ij}}\pi h(i_1,i_2,i_3,i_4) =$
$= \pi(-Z^2_{i_2 i_3}+Y_{i_2}Y_{i_3})$.

Hence $\tilde{y}(\frac{\delta}{\delta Z_{ij}}\pi h(i_1,i_2,i_3,i_4)) = \begin{cases} -1 \text{ if } \{i_1,i_4\}=\{i,j\} \text{ and } \{i_2,i_3\}=\{1,2\} \\ 0 \text{ in all other cases.} \end{cases}$

(4) Let $1 \leq i < j < k \leq n$.

If $t \in \{\pi f(i_1,i_2,i_3),\pi h(i_1,\ldots,i_4),\pi k(i_1,\ldots,i_5),\pi m(i_1,\ldots,i_6)\}$, $\frac{\delta t}{\delta W_{ijk}}$ belongs to $k[W_{123},\ldots,W_{n-2,n-1,n}]$ and therefore $\tilde{y}(\frac{\delta t}{\delta W_{ijk}}) = 0$.

We know (2.8) that $\pi g(\alpha,i_1,i_2,i_3) = \pm\pi g(\alpha,i_{\tau(1)},i_{\tau(2)},i_{\tau(3)})$ for all $\tau \in S_3$. So, in computing the rank of $J'(y)$, we can restrict ourselves to the elements $\pi g(i_1,i_2,i_3,i_4)$, $\pi g(i_2,i_1,i_3,i_4)$, $\pi g(i_3,i_1,i_2,i_4)$ and $\pi g(i_4,i_1,i_2,i_3)$ where $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$.

If $(i_1,i_2) \neq (1,2)$, Z_{12} does not occur among these elements, so certainly not in their partial derivatives with respect to W_{ijk}. On the other hand, if $(i_1,i_2) = (1,2)$, Z_{12} occurs only in $\pi g(1,2,i_3,i_4)$ and $\pi g(2,1,i_3,i_4)$.

$$\tilde{y}(\frac{\delta}{\delta W_{ijk}}\pi g(1,2,i_3,i_4)) = \begin{cases} 1 \text{ if } (i,j,k) = (1,i_3,i_4) \\ 0 \text{ in other cases} \end{cases}$$

$$\tilde{y}(\frac{\delta}{\delta W_{ijk}}\pi g(2,1,i_3,i_4)) = \begin{cases} 1 \text{ if } (i,j,k) = (2,i_3,i_4) \\ 0 \text{ in other cases.} \end{cases}$$

To conclude this computation notice that for all $3 \leq p < q < r \leq n$:

$$\tilde{y}(\frac{\delta}{\delta W_{ijk}}\pi l(p,q,r,1,2)) = \begin{cases} 1 \text{ if } (i,j,k) = (p,q,r) \\ 0 \text{ if } (i,j,k) \neq (p,q,r) \end{cases}$$

Collecting the results of (1), (2), (3) and (4) we obtain in case $n \geq 6$:

$$J'(y) = \begin{array}{l} \left.\begin{pmatrix} \theta & \alpha_1 & \theta & \theta \\ \theta & \theta & \theta & \alpha_3 \\ \theta & \theta & \alpha_2 & \theta \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \alpha_4 \\ \theta & \theta & \theta & \theta \end{pmatrix}\right. \end{array} \begin{array}{l} \} \text{ corresponding to the partial derivatives of } \pi f(i_1,i_2,i_3). \\ \} \ \pi g(i_1,i_2,i_3,i_4) \\ \} \ \pi h(i_1,i_2,i_3,i_4) \\ \} \ \pi k(i_1,i_2,i_3,i_4,i_5) \\ \} \ \pi l(i_1,i_2,i_3,i_4,i_5) \\ \} \ \pi m(i_1,i_2,i_3,i_4,i_5,i_6) \end{array}$$

Hence rank $J'(y) = \text{rank } \alpha_1 + \text{rank } \alpha_2 + \text{rank} \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}$

We know that $\pi f(i_1,i_2,i_3) = \pi f(i_{\tau(1)},i_{\tau(2)},i_{\tau(3)})$ for all $\tau \in S_3$ (2.8.). Let $f_p = \pi f(1,2,p)$ where $3 \leq p \leq n$, then it follows from (2) above that

$$\text{rank } \alpha_1 = \text{rank} \begin{pmatrix} \frac{\partial f_3}{\partial Y_3} & \cdots & \frac{\partial f_3}{\partial Y_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial Y_3} & \cdots & \frac{\partial f_n}{\partial Y_n} \end{pmatrix}(y) = \text{rank } (-I_{n-2}) = n-2.$$

$\pi h(i_1,i_2,i_3,i_4) = \pi h(i_1,i_3,i_2,i_4) = \pi h(i_4,i_2,i_3,i_1)$. In view of (3) the rank of α_2 is fully determined by the partial derivatives of the elements $\pi h(p,1,2,q)$ with $3 \leq p < q \leq n$. Now

$$\tilde{y}\left(\frac{\delta}{\delta Z_{ij}} \pi h(p,1,2,q)\right) = \begin{cases} -1 \text{ if } (p,q) = (i,j) \\ 0 \text{ if } (p,q) \neq (i,j) \end{cases}$$

So rank $\alpha_2 = \binom{n-2}{2}$.

From (4) it follows that rank $\begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix} \geq \text{rank} \begin{pmatrix} \beta_3 \\ \beta_4 \end{pmatrix}$ where β_3 corresponds to the partial derivatives of $\pi g(1,2,p,q)$ and $\pi g(2,1,p,q)$ with respect to W_{ijk} $(3 \leq p < q \leq n)$ and β_4 to those of $\pi l(p,q,r,1,2)$ $(3 \leq p < q < r \leq n)$.

Now notice that $\tilde{y}(\frac{\partial}{\partial W_{ijk}} \pi l(p,q,r,1,2)) = 0$ if $i \in \{1,2\}$ and $\tilde{y}(\frac{\delta}{\delta W_{ijk}} \pi g(1,2,p,q)) = \tilde{y}(\frac{\delta}{\delta W_{ijk}} \pi g(2,1,p,q)) = 0$ if $i \notin \{1,2\}$.

Hence rank $\begin{pmatrix}\beta_3\\ \beta_4\end{pmatrix}$ = rank β_3 + rank β_4. Furthermore it is easy to see that rank $\beta_3 = 2\binom{n-2}{2}$ and rank $\beta_4 = \binom{n-2}{3}$.

We conclude:

If $n \geq 6$, rank $J'(y) \geq n-2+\binom{n-2}{2}+2\binom{n-2}{2}+\binom{n-2}{3} = \binom{n}{3}+\frac{1}{2}(n-2)(n-3)$.

Because the $m(i_1,\dots,i_6)$ play no part in the above computation the result is also true if $n = 5$.

If $n = 4$ we have $\binom{n}{3}+\frac{1}{2}(n-2)(n-3) = 5$. The ideal J contains $a_1 = \pi f(1,2,3)$, $a_2 = \pi f(1,2,4)$, $a_3 = \pi g(1,2,3,4)$, $a_4 = \pi g(2,1,3,4)$ and $a_5 = \pi h(3,1,2,4)$.

Hence

$$\text{rank } J'(y) \geq \begin{pmatrix} \frac{\delta a_1}{\delta Y_3} & \frac{\delta a_1}{\delta Y_4} & \frac{\delta a_1}{\delta Z_{34}} & \frac{\delta a_1}{\delta W_{134}} & \frac{\delta a_1}{\delta W_{234}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\delta a_5}{\delta Y_3} & \frac{\delta a_5}{\delta Y_4} & \frac{\delta a_5}{\delta Z_{34}} & \frac{\delta a_5}{\delta W_{134}} & \frac{\delta a_5}{\delta W_{234}} \end{pmatrix}(y) = \text{rank} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} = 5.$$

Finally if $n = 3$, $\binom{n}{3}+\frac{1}{2}(n-2)(n-3) = 1$ and J is generated by $f = \pi f(1,2,3)$; rank $J'(y)$ = rank $(\tilde{y}(\frac{\delta f}{\delta Y_1})\ \tilde{y}(\frac{\delta f}{\delta Y_2})\ \tilde{y}(\frac{\delta f}{\delta Y_3}))$ = rank $(0\ 0\ -1) = 1$.

QED

<u>corollary 2.17</u>. $\delta J = J$.

<u>proof</u>:

$t \in J \Rightarrow t \in \ker \alpha \Rightarrow \alpha\delta(t) = d\alpha(t) = 0$ (cf. 1.7.) $\Rightarrow \delta t \in \ker \alpha = J$.

So $\delta J \subset J$. Since $\delta^2 = 1$ it then follows that $\delta J = J$.

§3. Y_S is the smooth part of Y ($n \geq 3$).

In view of III, §2 we restrict ourselves to the cases $n \geq 4$. Furthermore we know $Y = \operatorname{Spec} {}^{\Sigma}/J = V(J)$ because $J = \ker \alpha$ (§2). Putting it this way, we also have $Y - Y_S = V(R) \subset V(J)$ (cf. 2.4 and 2.5) where R is the ideal generated by: $Z_{ij}^2 - Y_i Y_j \quad (1 \leq i < j \leq n)$

$$\left.\begin{array}{l} W_{ijk} \\ Z_{ij}Z_{ik} - Y_i Z_{jk} \\ Z_{ij}Z_{jk} - Y_j Z_{ik} \\ Z_{ik}Z_{jk} - Y_k Z_{ij} \end{array}\right\} \quad (1 \leq i < j < k \leq n)$$

In the proof of lemma 2.4 we have shown that $\pi(Z_{ij}Z_{kl} - Z_{pq}Z_{rs}) \in \sqrt{R}$ whenever $\{p,q,r,s\} = \{i,j,k,l\}$.

<u>theorem 3.1</u>. If $y \in Y - Y_S$ then Y is <u>not</u> smooth at y.

<u>proof</u>:

Let $y = (x_1,\dots,x_n; y_1,\dots,y_n; z_{12},\dots,z_{n-1,n}; 0,\dots,0) \in V(R)$. Let J be the Jacobi matrix belonging to J, i.e.,

$$J = \left(\frac{\delta t}{\delta X_1} \cdots \frac{\delta t}{\delta X_n} \;\; \frac{\delta t}{\delta Y_1} \cdots \frac{\delta t}{\delta Y_n} \;\; \frac{\delta t}{\delta Z_{12}} \cdots \frac{\delta t}{\delta Z_{n-1,n}} \;\; \frac{\delta t}{\delta W_{123}} \cdots \frac{\delta t}{\delta W_{n-2,n-1,n}} \right)$$

where t runs through the generators of J given in 1.8.

Now: Y is <u>not</u> smooth at y iff $\operatorname{rank} J(y) < \binom{n}{3} + \frac{1}{2}(n-2)(n-3)$ [8].

We prove that $\frac{\delta t}{\delta X_i}$ $(1 \leq i \leq n)$, $\frac{\delta t}{\delta Y_i}$ $(1 \leq i \leq n)$ and $\frac{\delta t}{\delta Z_{ij}}$ $(1 \leq i < j \leq n)$ belong to $\sqrt{R}$ if $t \in \{\pi f(i_1,i_2,i_3), \pi g(i_1,\dots,i_4), \pi h(i_1,\dots,i_4), \pi k(i_1,\dots,i_5), \pi l(i_1,\dots,i_5), \pi m(i_1,\dots,i_6)\}$.

If this is proved we are ready because then it follows that

$$\operatorname{rank} J(y) = \operatorname{rank} \left(\frac{\delta t}{\delta W_{123}} \cdots \frac{\delta t}{\delta W_{n-2,n-1,n}} \right)(y) \leq \binom{n}{3} < \binom{n}{3} + \tfrac{1}{2}(n-2)(n-3)$$

(remember $n \geq 4$)

(1) $\frac{\delta t}{\delta X_i} = 0$ for all t mentioned above.

(2) $\frac{\delta}{\delta Y_i}\pi f(i_1,i_2,i_3) = \delta_{ii_1}[\pi(Y_{i_2}Y_{i_3}-Z^2_{i_2i_3})]+\delta_{ii_2}[\pi(Y_{i_1}Y_{i_3}-Z^2_{i_1i_3})]+$

$$+\delta_{ii_3}[\pi(Y_{i_1}Y_{i_2}-Z^2_{i_1i_2})] \in R$$

$$\frac{\delta}{\delta Y_i}\pi g(i_1,i_2,i_3,i_4) = -\delta_{ii_1}[\pi W_{i_2i_3i_4}] \in R$$

$$\frac{\delta}{\delta Y_i}\pi h(i_1,i_2,i_3,i_4) = -\delta_{ii_2}[\pi(Z_{i_1i_3}Z_{i_3i_4}-Y_{i_3}Z_{i_1i_4})]+$$

$$-\delta_{ii_3}[\pi(Z_{i_1i_2}Z_{i_2i_4}-Y_{i_2}Z_{i_1i_4})] \in R$$

$$\frac{\delta}{\delta Y_i}\pi k(i_1,\ldots,i_5) = \delta_{ii_1}[\pi(Z_{i_2i_4}Z_{i_3i_5}-Z_{i_2i_5}Z_{i_3i_4})] \in \sqrt{R}$$

$$\frac{\delta}{\delta Y_i}\pi l(i_1,\ldots,i_5) = \frac{\delta}{\delta Y_i}\ \pi m(i_1,\ldots,i_6) = 0$$

(3) $\frac{\delta}{\delta Z_{ij}}\pi f(i_1,i_2,i_3)$ belongs to the subideal of R generated by

$\pi(Z_{i_1i_3}Z_{i_2i_3}-Y_{i_3}Z_{i_1i_2})$, $\pi(Z_{i_1i_2}Z_{i_2i_3}-Y_{i_2}Z_{i_1i_3})$ and

$\pi(Z_{i_1i_2}Z_{i_1i_3}-Y_{i_1}Z_{i_2i_3})$.

$$\frac{\delta}{\delta Z_{ij}}\pi g(i_1,\ldots,i_4) \in\ k[W_{123},\ldots,W_{n-2,n-1,n}] \subset R.$$

$$\frac{\delta}{\delta Z_{ij}}\pi h(i_1,i_2,i_3,i_4)=\begin{cases}\pi(Z_{i_2i_3}Z_{i_3i_4}-Y_{i_3}Z_{i_2i_4})\in R & \text{if } Z_{ij}=\pi Z_{i_1i_2}\\ \pi(Z_{i_2i_3}Z_{i_2i_4}-Y_{i_2}Z_{i_3i_4})\in R & \text{if } Z_{ij}=\pi Z_{i_1i_3}\\ \pi(-Z^2_{i_2i_3}+Y_{i_2}Y_{i_3})\in R & \text{if } Z_{ij}=\pi Z_{i_1i_4}\\ \pi(Z_{i_1i_3}Z_{i_2i_4}+Z_{i_1i_2}Z_{i_3i_4}-2Z_{i_1i_4}Z_{i_2i_3})\in\sqrt{R} & \text{if } Z_{ij}\ \pi Z_{i_2i_3}\\ \pi(Z_{i_1i_3}Z_{i_2i_3}-Y_{i_3}Z_{i_1i_2})\in R & \text{if } Z_{ij}=\pi Z_{i_2i_4}\\ \pi(Z_{i_1i_2}Z_{i_2i_3}-Y_{i_2}Z_{i_1i_3})\in R & \text{if } Z_{ij}=\pi Z_{i_3i_4}\\ 0 & \text{in all other cases}\end{cases}$$

$$\frac{\delta}{\delta Z_{ij}}\pi k(i_1,\dots,i_5)=\begin{cases}\pi(Z_{i_1i_5}Z_{i_3i_4}-Z_{i_1i_4}Z_{i_3i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_2}\\ \pi(Z_{i_1i_4}Z_{i_2i_5}-Z_{i_2i_4}Z_{i_1i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_3}\\ \pi(Z_{i_1i_3}Z_{i_2i_5}-Z_{i_1i_2}Z_{i_3i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_4}\\ \pi(Z_{i_1i_2}Z_{i_3i_4}-Z_{i_1i_3}Z_{i_2i_4})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_5}\\ \pi(-Z_{i_1i_3}Z_{i_1i_5}+Y_{i_1}Z_{i_3i_5})\in\mathcal{R} & \text{if } Z_{ij}=\pi Z_{i_2i_4}\\ \pi(Z_{i_1i_3}Z_{i_1i_4}-Y_{i_1}Z_{i_3i_4})\in\mathcal{R} & \text{if } Z_{ij}=\pi Z_{i_2i_5}\\ \pi(Z_{i_1i_2}Z_{i_1i_5}-Y_{i_1}Z_{i_2i_5})\in\mathcal{R} & \text{if } Z_{ij}=\pi Z_{i_3i_4}\\ \pi(-Z_{i_1i_2}Z_{i_1i_4}+Y_{i_1}Z_{i_2i_4})\in\mathcal{R} & \text{if } Z_{ij}=\pi Z_{i_3i_5}\\ 0 & \text{in all other cases}\end{cases}$$

$$\frac{\delta}{\delta Z_{ij}}\pi l(i_1,\dots,i_5)\in k[W_{123},\dots,W_{n-2,n-1,n}]\subset\mathcal{R}$$

$$\frac{\delta}{\delta Z_{ij}}\pi m(i_1,\dots,i_6)=\begin{cases}\pi(Z_{i_3i_5}Z_{i_4i_6}-Z_{i_3i_6}Z_{i_4i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_2}\\ \pi(Z_{i_2i_5}Z_{i_3i_6}-Z_{i_3i_5}Z_{i_2i_6})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_4}\\ \pi(Z_{i_2i_6}Z_{i_3i_4}-Z_{i_2i_3}Z_{i_4i_6})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_5}\\ \pi(Z_{i_2i_3}Z_{i_4i_5}-Z_{i_2i_5}Z_{i_3i_4})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_1i_6}\\ \pi(Z_{i_1i_6}Z_{i_4i_5}-Z_{i_1i_5}Z_{i_4i_6})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_2i_3}\\ \pi(Z_{i_1i_4}Z_{i_3i_6}-Z_{i_1i_6}Z_{i_3i_4})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_2i_5}\\ \pi(Z_{i_1i_5}Z_{i_3i_4}-Z_{i_1i_4}Z_{i_3i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_2i_6}\\ \pi(Z_{i_1i_5}Z_{i_2i_6}-Z_{i_1i_6}Z_{i_2i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_3i_4}\\ \pi(Z_{i_1i_2}Z_{i_4i_6}-Z_{i_1i_4}Z_{i_2i_6})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_3i_5}\\ \pi(Z_{i_1i_4}Z_{i_2i_5}-Z_{i_1i_2}Z_{i_4i_5})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_3i_6}\\ \pi(Z_{i_1i_6}Z_{i_2i_3}-Z_{i_1i_2}Z_{i_3i_6})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_4i_5}\\ \pi(Z_{i_1i_2}Z_{i_3i_5}-Z_{i_1i_5}Z_{i_2i_3})\in\sqrt{\mathcal{R}} & \text{if } Z_{ij}=\pi Z_{i_4i_6}\\ 0 & \text{in all other cases}\end{cases}$$

QED.

<u>lemma 3.2</u>. Let $f : X \to Y$ be a surjective morphism of k-schemes of finite type. Suppose X is smooth and f is a smooth morphism of relative dimension $r = \dim X - \dim Y$. Then Y is also smooth.

<u>proof</u>: Let y be a closed point of Y and x a closed point of X such that $f(x) = y$. Because of the smoothness of f we have an exact sequence of $\mathcal{O}_X$-modules ([6]) : $0 \to f^*(\Omega_Y) \to \Omega_X \to \Omega_{X/Y} \to 0$ (1)

Therefore we also have the exact sequence of $\mathcal{O}_{X,x}$-modules:

$$0 \to f^*(\Omega_Y)_x \to \Omega_{X,x} \to \Omega_{X/Y,x} \to 0 \quad (2)$$

Now $f^*(\Omega_Y)_x \simeq \Omega_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. Tensoring (2) over $\mathcal{O}_{X,x}$ with $\kappa(x)$ ($\simeq k$) we obtain an exact sequence:

$$\mathrm{Tor}_1^{\mathcal{O}_{X,x}}(\Omega_{X/Y,x};\kappa(x)) \to (\Omega_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to \Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to 0$$

Since $\Omega_{X/Y,x}$ is a free $\mathcal{O}_{X,x}$-module of rank r [6], $\mathrm{Tor}_1^{\mathcal{O}_{X,x}}(\Omega_{X/Y,x};\kappa(x))=0$.

Furthermore $(\Omega_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} \kappa(x) \simeq \Omega_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ and therefore we obtain the following exact sequence:

$$0 \to \Omega_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) \to \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to \Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to 0 \quad (3)$$

Now $T(X)_x \simeq \mathrm{Hom}_{k\text{-mod}}(\Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x),k)$, so from (3) we deduce the exact sequence

$$0 \to \mathrm{Hom}_k(\Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x),k) \to T(X)_x \to T(Y)_y \to 0 \quad (4)$$

Since X is smooth, $\dim T(X)_x = \dim X$. The dimension of $\mathrm{Hom}_k(\Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x),k)$ equals the dimension of the tangent space of $f^{-1}(y)$ at x, which, in turn, equals $r = \dim X - \dim Y$, because f is smooth of relative dimension r.

Using the exactness of (4) it follows that $\dim T(Y)_y = \dim Y$ or equivalently, Y is smooth at y.

We want to apply this lemma to $\phi' : X_S \to Y_S$.

$X \simeq \mathbb{A}^{4n}$ is smooth, X_S is an open subset of X and hence X_S is smooth too. Since $Y_S = \phi'(X_S)$, $\phi' : X_S \to Y_S$ is a surjective morphism.

$$\begin{tikzcd} X_S \arrow[r, hook] \arrow[d, "\phi'"'] & X = \operatorname{Spec} R \arrow[d, "\phi"] \\ Y_S \arrow[r, hook] & Y = \operatorname{Spec} R \end{tikzcd}$$

ϕ, and also ϕ', is a morphism of finite type, for $k \subset R^G \subset R$, R is a k-algebra of finite type and therefore R is a R^G-algebra of finite type. Hence $\phi' : X_S \to Y_S$ is a surjective morphism of k-schemes of finite type.

I claim that ϕ' is smooth of relative dimension 3.

<u>proposition 3.3</u>. $\phi : X \to Y$ is a flat morphism of finite type.

<u>proof</u>:

ϕ is a flat morphism iff R is a flat R^G-module. Using [10] we have:

$R = R_G \oplus R^G$

The Reynoldsoperator $E : R \to R$, i.e.,the projection on R^G, has the property: if $x \in R^G$, $y \in R$ then $E(xy) = xE(y)$.

In particular R_G is a R^G-module.

Let M be a R^G-module. We have to show that $M \to R \otimes_{R^G} M$ is injective. But $M \to R \otimes_{R^G} M = (R^G \oplus R_G) \otimes_{R^G} M \xrightarrow{\sim} M \oplus (R_G \otimes_{R^G} M)$. The composition maps $m \in M$ onto $(m,0)$ and therefore $M \to R \otimes_{R^G} M$ is injective.

<u>corollary 3.4</u>. $\phi' : X_S \to Y_S$ is a flat morphism of finite type.

<u>proof</u>:

Flatness is preserved under base changing and the adjacent diagram is commutative

$$\begin{tikzcd} X_S \simeq X \times_Y Y_S \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[d, "\phi'"', bend right] & X \arrow[d, "\phi"] \\ Y_S \arrow[r, hook] & Y \end{tikzcd}$$

Finally we want to apply the following theorem (cf. [8]):

theorem. Let $f : X \to Y$ be a morphism of finite type. Then f is smooth of relative dimension $r = \dim X - \dim Y$ if and only if f is flat and its geometric fibres are disjoint unions of r-dimensional smooth varieties.

Dim $X_S = \dim X = 4n$, $\dim Y_S = \dim Y = 4n-3$ (cf. 2.15) and $\phi' : X_S \to Y_S$ is a flat morphism of finite type. If y is a closed point of Y_S, its geometric fibre is isomorphic with $PGl(2,k)$, so smooth of dimension 3.

Combining the above results we have proved the smoothness of Y_S.

theorem 3.5. Y_S is the smooth part of Y ($n \geq 3$).

proof: Y_S is smooth and an open subset of Y. Hence Y is smooth at the points of Y_S. On the other hand Y is not smooth at the points outside Y_S (3.1). Therefore Y_S is the smooth part of Y.

Chapter V. The study of the singular locus of Y.

§1. The dimension of $Y\text{-}Y_S$; $Y\text{-}Y_S \simeq \mathbb{A}^n * \mathbb{A}^n$.

definition 1.1. Let $A = k[X_1,\dots,X_n,Y_1,\dots,Y_n]$ be the ring of coordinates of the variety $\mathbb{A}^n \times \mathbb{A}^n$. Let $\varepsilon : \mathbb{A}^n \times \mathbb{A}^n \to X = \operatorname{Spec} R$ be the morphism induced by the k-algebra homomorphism $\varepsilon^* : R \to A$ such that, for all $1 \leq i \leq n$:

$$\varepsilon^*(X_{i;11}) = X_i,\ \varepsilon^*(X_{i;22}) = Y_i \text{ and } \varepsilon^*(X_{i;12}) = \varepsilon^*(X_{i;21}) = 0.$$

Restricting ε to the sets of closed points one obtains a map from k^{2n} to $M(n,k)^2$ given by $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n) \mapsto (\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix})$.

proposition 1.2. $\phi \circ \varepsilon : \mathbb{A}^n \times \mathbb{A}^n \to Y$ maps $\mathbb{A}^n \times \mathbb{A}^n$ onto $Y\text{-}Y_S$.

proof: Let $x \in \mathbb{A}^n \times \mathbb{A}^n$ be a closed point corresponding to $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n) \in k^{2n}$. Then $\varepsilon(x)$ corresponds to $(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix})$. Hence $\varepsilon(x) \notin X_S$ (cf. II.3.2.) and therefore $\phi\varepsilon(x) \in Y\text{-}Y_S$.

Let $y \in Y\text{-}Y_S$ be a closed point. According to theorem I.2.6. $\phi^{-1}(y)$ contains a (unique) closed orbit. Since $y \notin Y_S$ and $Y_S = \phi(X_S)$ we conclude from II.3.2. that it has to be the orbit of a closed point corresponding to a diagonalizable element of $M(2,k)^n$. Hence $\phi^{-1}(y)$ contains a closed point x' which corresponds to an element of the form $(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix})$. Let $x \in \mathbb{A}^n \times \mathbb{A}^n$ be the closed point belonging to $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)$, then obviously $\phi\varepsilon(x) = y$.

From the proposition above it follows that there exists a surjective morphism ψ such that the diagram below commutes.

$$\begin{tikzcd} \mathbb{A}^n \times \mathbb{A}^n \arrow[r, "\varepsilon"] \arrow[d, "\psi"'] & X \arrow[d, "\phi"] \\ Y\text{-}Y_S \arrow[r, hook] & Y \end{tikzcd}$$

Putting $Y = \operatorname{Spec} {}^{\Sigma}/J$ and $Y\text{-}Y_S = \operatorname{Spec} {}^{\Sigma}/\sqrt{R}$ (cf. the beginning of IV, §3), this corresponds to the commutativity of the following diagram:

$$\begin{tikzcd} A = k[X_1,\dots,X_n,Y_1,\dots,Y_n] & & R \arrow[ll, "\varepsilon^*"'] & \\ {}^{\Sigma}/\sqrt{R} \arrow[u, "\psi^*"] & & {}^{\Sigma}/J \arrow[ll] \arrow[u, "\bar{\alpha}"'] & \Sigma \arrow[l, "p"] \arrow[ul, "\alpha"'] \\ g+\sqrt{R} & & g+J \arrow[ll, mapsto] & \end{tikzcd}$$

Explicitly we have:

$$\begin{aligned} 1.3:\quad \psi^*(X_i+\sqrt{R}) &= \varepsilon^*\bar{\alpha}(X_i+J) = \varepsilon^*\alpha(X_i) = \varepsilon^*(\mathrm{Tr}\underline{X}_i) = X_i+Y_i && (1\leq i\leq n) \\ \psi^*(Y_i+\sqrt{R}) &= \varepsilon^*(2\mathrm{Tr}\underline{X}_i^2-(\mathrm{Tr}\underline{X}_i)^2) = (X_i-Y_i)^2 && (1\leq i\leq n) \\ \psi^*(Z_{ij}+\sqrt{R}) &= \varepsilon^*(2\mathrm{Tr}\underline{X}_i\underline{X}_j-\mathrm{Tr}\underline{X}_i\mathrm{Tr}\underline{X}_j) = (X_i-Y_i)(X_j-Y_j) && (1\leq i<j\leq n) \\ \psi^*(W_{ijk}+\sqrt{R}) &= 0 && (1\leq i<j<k\leq n) \end{aligned}$$

<u>proposition 1.4.</u> $Y\text{-}Y_S$ is irreducible and $\dim(Y\text{-}Y_S) = 2n$.

<u>proof</u>: $\psi : \mathbb{A}^n \times \mathbb{A}^n \to Y\text{-}Y_S$ is surjective, $\mathbb{A}^n \times \mathbb{A}^n$ is irreducible and so $Y\text{-}Y_S$ is irreducible too. We claim that $\dim \psi^{-1}(y) = 0$ for all closed points $y \in Y\text{-}Y_S$. It then follows immediately that $\dim(Y\text{-}Y_S) = \dim \mathbb{A}^n \times \mathbb{A}^n = 2n$.

Let x and x' be closed points of $\mathbb{A}^n \times \mathbb{A}^n$ corresponding to $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)$ and $(\alpha_1',\dots,\alpha_n',\beta_1',\dots,\beta_n')$ respectively. Then $\psi(x) = \psi(x')$ iff $i(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix}) = i(\begin{pmatrix}\alpha_1' & 0\\ 0 & \beta_1'\end{pmatrix},\dots,\begin{pmatrix}\alpha_n' & 0\\ 0 & \beta_n'\end{pmatrix})$, which - by definition of i - is equivalent to

$$\begin{cases} (1)\ \alpha_i+\beta_i = \alpha'_i+\beta'_i & (1 \leq i \leq n) \\ (2)\ (\alpha_i-\beta_i)^2 = (\alpha'_i-\beta'_i)^2 & (1 \leq i \leq n) \\ (3)\ \alpha_i\alpha_j+\beta_i\beta_j = \alpha'_i\alpha'_j+\beta'_i\beta'_j & (1 \leq i < j \leq n) \end{cases}$$

Now distinguish three cases:

<u>Case 1</u>: $\alpha_i = \beta_i$ for all $1 \leq i \leq n$. Then (1) implies $\alpha_i = \alpha'_i = \beta_i = \beta'_i$, so $x = x'$.

<u>Case 2</u>: $\alpha_p \neq \beta_p$ for some p and $\alpha_p-\beta_p = \alpha'_p-\beta'_p$.
Since $\alpha_p+\beta_p = \alpha'_p+\beta'_p$ and $\alpha_p-\beta_p = \alpha'_p-\beta'_p$ we have $\alpha_p = \alpha'_p$ and $\beta_p = \beta'_p$.
From (1) and (3) it follows that $\begin{pmatrix} 1 & 1 \\ \alpha_p & \beta_p \end{pmatrix}\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha_p & \beta_p \end{pmatrix}\begin{pmatrix} \alpha'_i \\ \beta'_i \end{pmatrix}$ for all $i \neq p$.
Now $\begin{pmatrix} 1 & 1 \\ \alpha_p & \beta_p \end{pmatrix}$ is invertible and therefore $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$ for all i, i.e., $x = x'$.

<u>Case 3</u>: $\alpha_p \neq \beta_p$ for some p and $\alpha_p-\beta_p = -(\alpha'_p-\beta'_p)$.
From (1) we deduce: $\alpha'_p = \beta_p$ and $\beta'_p = \alpha_p$. Combining this with (3) we obtain $\begin{pmatrix} 1 & 1 \\ \alpha_p & \beta_p \end{pmatrix}\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha_p & \beta_p \end{pmatrix}\begin{pmatrix} \beta'_i \\ \alpha'_i \end{pmatrix}$ if $i \neq p$. Hence $\alpha'_i = \beta_i$, $\alpha_i = \beta'_i$ for all i.

So we even proved that $\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix},\ldots,\begin{pmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{pmatrix}\right)$ and $\left(\begin{pmatrix} \alpha'_1 & 0 \\ 0 & \beta'_1 \end{pmatrix},\ldots,\begin{pmatrix} \alpha'_n & 0 \\ 0 & \beta'_n \end{pmatrix}\right)$ have the same invariants iff
$(\alpha'_1,\ldots,\alpha'_n,\beta'_1,\ldots,\beta'_n)\in\{(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n),(\beta_1,\ldots,\beta_n,\alpha_1,\ldots,\alpha_n)\}$
which obviously implies that $\dim \psi^{-1}(y) = 0$. QED.

The proof of the proposition up here suggest to examine the question whether $Y-Y_S$ is isomorphic with $\mathbb{A}^n * \mathbb{A}^n$, the symmetric product of $\mathbb{A}^n$ with itself. Evidently $\mathbb{A}^n * \mathbb{A}^n$ is a quotient for the natural action of S_2 on $\mathbb{A}^n \times \mathbb{A}^n$.

<u>definition 1.5</u>. Let B denote the subalgebra of A consisting of the elements $f(X_1,\ldots,X_n,Y_1,\ldots,Y_n)$ such that

$f(X_1,\ldots,X_n,Y_1,\ldots,Y_n) = f(Y_1,\ldots,Y_n,X_1,\ldots,X_n)$. Clearly $\mathbb{A}^n * \mathbb{A}^n = \operatorname{Spec} B$.

Notice that $X_i+Y_i \in B$ $(1 \le i \le n)$ and $(X_i-Y_i)(X_j-Y_j) \in B$ $(1 \le i \le j \le n)$. Hence in view of 1.3 we have $\psi^*({}^{\Sigma}/\sqrt{R}) \subset B$. In order to prove $\psi^*({}^{\Sigma}/\sqrt{R}) = B$ we need a lemma.

<u>lemma 1.6</u>. Let S be a commutative ring with an identity element in which 2 is a unit. Let $r \in \mathbb{N}$, $A_i \in M(2,S)$ for $1 \le i \le r$ and $A_0 = I$. Then $\operatorname{Tr}A_1\ldots A_r$ can be written as a polynomial in the elements of the set

$$\mathcal{D} = \{\operatorname{Tr}A_iA_jA_k \mid i,j,k \in \{0,1,\ldots,r\}\}$$

with coefficients in $\mathbb{Z}[\frac{1}{2}]$.

<u>proof</u>: Evidently $\operatorname{Tr}A_1\ldots A_r = \operatorname{Tr}A_{i_1}\ldots A_{i_r}$ if $(i_1,\ldots,i_r)$ is deduced from $(1,\ldots,r)$ by cyclic permutation. By direct computation one verifies:

If $F(X_1,X_2,X_3) = \operatorname{Tr}X_1X_2X_3 + \operatorname{Tr}X_1X_3X_2 - \operatorname{Tr}X_1\operatorname{Tr}X_2X_3 - \operatorname{Tr}X_2\operatorname{Tr}X_1X_3 - \operatorname{Tr}X_3\operatorname{Tr}X_1X_2 + \operatorname{Tr}X_1\operatorname{Tr}X_2\operatorname{Tr}X_3$, then $F(A,B,C) = 0$ for all $A,B,C \in M(2,S)$ (cf. I. §1).

Now the lemma is proved by induction with respect to r using: If $r \ge 4$ and $E = A_4\ldots A_m$ then

$$\begin{aligned}
2\operatorname{Tr}A_1\ldots A_m &= \operatorname{Tr}A_1A_2A_3E + \operatorname{Tr}A_3A_1A_2E - (\operatorname{Tr}A_3A_1A_2E + \operatorname{Tr}A_2A_3A_1E) + \operatorname{Tr}A_2A_3A_1E + \operatorname{Tr}A_1A_2A_3E \\
&= F(A_1A_2,A_3,E) - F(A_3A_1,A_2,E) + F(A_2A_3,A_1,E) + \\
&\quad + \operatorname{Tr}A_1\operatorname{Tr}A_2A_3E - \operatorname{Tr}A_2\operatorname{Tr}A_3A_1E + \operatorname{Tr}A_3\operatorname{Tr}A_1A_2E + \\
&\quad + \operatorname{Tr}A_1A_2\operatorname{Tr}A_3E - \operatorname{Tr}A_1A_3\operatorname{Tr}A_2E + \operatorname{Tr}A_2A_3\operatorname{Tr}A_1E + \\
&\quad + \operatorname{Tr}E[\operatorname{Tr}A_1A_2A_3 - \operatorname{Tr}A_1\operatorname{Tr}A_2A_3 + \operatorname{Tr}A_2\operatorname{Tr}A_1A_3 - \operatorname{Tr}A_3\operatorname{Tr}A_1A_2].
\end{aligned}$$

<u>proposition 1.7</u>. The k-algebra B (defined in 1.5.) is generated by the elements X_i+Y_i and $(X_i-Y_i)(X_j-Y_j)$ where $1 \le i \le n$, $1 \le j \le n$. Obviously this implies $B = \psi^*({}^{\Sigma}/\sqrt{R})$.

<u>proof</u>: It is easy to see that f belongs to B iff f is a k-linear combination of terms of the type $X_1^{\alpha_1}\ldots X_n^{\alpha_n}Y_1^{\beta_1}\ldots Y_n^{\beta_n}+X_1^{\beta_1}\ldots X_n^{\beta_n}Y_1^{\alpha_1}\ldots Y_n^{\alpha_n}$.

Now $X_1^{\alpha_1}\ldots X_n^{\alpha_n}Y_1^{\beta_1}\ldots Y_n^{\beta_n}+X_1^{\beta_1}\ldots X_n^{\beta_n}Y_1^{\alpha_1}\ldots Y_n^{\alpha_n} =$

$$= \mathrm{Tr}\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}^{\alpha_1}\cdots\begin{pmatrix} X_n & 0 \\ 0 & Y_n \end{pmatrix}^{\alpha_n}\begin{pmatrix} Y_1 & 0 \\ 0 & X_1 \end{pmatrix}^{\beta_1}\cdots\begin{pmatrix} Y_n & 0 \\ 0 & X_n \end{pmatrix}^{\beta_n}.$$

Applying lemma 1.6 with $S = A = k[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$ we can write such a term as a polynomial (with coefficients in k) in the following elements of B:

a) $\mathrm{Tr}\begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} Y_i & 0 \\ 0 & X_i \end{pmatrix} = X_i+Y_i \quad (1\le i\le n)$

b) $\mathrm{Tr}\begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix}\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} Y_i & 0 \\ 0 & X_i \end{pmatrix}\begin{pmatrix} Y_j & 0 \\ 0 & X_j \end{pmatrix} = X_iX_j+Y_iY_j \quad (1\le i\le n,\ 1\le j\le n)$

c) $\mathrm{Tr}\begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix}\begin{pmatrix} Y_j & 0 \\ 0 & X_j \end{pmatrix} = X_iY_j+X_jY_i \quad (1\le i\le n,\ 1\le j\le n)$

d) $\mathrm{Tr}B_1B_2B_3$ with $B_i \in \{\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix},\ldots,\begin{pmatrix} X_n & 0 \\ 0 & Y_n \end{pmatrix},\begin{pmatrix} Y_1 & 0 \\ 0 & X_1 \end{pmatrix},\ldots,\begin{pmatrix} Y_n & 0 \\ 0 & X_n \end{pmatrix}\}$

We can drop the elements listed under d) because in general:

$$2\mathrm{Tr}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\mathrm{Tr}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}-\mathrm{Tr}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\mathrm{Tr}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & e \end{pmatrix}+$$

$$+\mathrm{Tr}\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\mathrm{Tr}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

The proof can be finished by remarking that for all $1 \le i \le n$ and $1 \le j \le n$:

$$2(X_iX_j+Y_iY_j) = (X_i+Y_i)(X_j+Y_j)+(X_i-Y_i)(X_j-Y_j)$$
$$2(X_iY_j+X_jY_i) = (X_i+Y_i)(X_j+Y_j)-(X_i-Y_i)(X_j-Y_j)$$

<u>definition 1.8</u>. The following corresponding diagrams are commutative, which defines η.

$$\begin{tikzcd} \mathbb{A}^n\times\mathbb{A}^n \arrow[d] \arrow[dr, "\psi"] & \\ \mathbb{A}^n*\mathbb{A}^n \arrow[r, "\eta"'] & Y-Y_S \end{tikzcd} \qquad \begin{tikzcd} A & \\ B \arrow[u, hook] & \Sigma/\sqrt{R} \arrow[l, "\eta^*"] \arrow[ul, "\psi^*"'] \end{tikzcd}$$

<u>proposition 1.9</u>. η^* is an isomorphism and therefore $\eta : \mathbb{A}^n * \mathbb{A}^n \overset{\sim}{\to} Y-Y_S$.

<u>proof</u>: From 1.3 and 1.7 it is clear that η^* is surjective. The injectivity of η^* remains to be proved.

From IV.2.5. we know that $x \in V(R)$ iff there exists a diagonalizable element A in $M(2,k)^n$ such that $i(A) = x$. Notice also that

$i(\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix}) = (..\alpha_i+\beta_i..;..(\alpha_i-\beta_i)^2..;..(\alpha_i-\beta_i)(\alpha_j-\beta_j)..;0,..,0)$.

Now suppose $g = g(..X_i..;..Y_i..;..Z_{ij}..;..W_{ijk}..) \in \Sigma$ and $g+\sqrt{R} \in \ker \eta^*$. Then $g(..X_i+Y_i..;..(X_i-Y_i)^2..;..(X_i-Y_i)(X_j-Y_j)..;0,\dots,0) = 0$. In view of the above remark this is equivalent to: $\widetilde{i(A)}(g) = 0$ for all $A \in M(2,k)^n$ of the form $A = (\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_n\end{pmatrix},\dots,\begin{pmatrix}\alpha_n & 0\\ 0 & \beta_n\end{pmatrix})$. But then $\widetilde{i(A)}(g) = 0$ for all diagonalizable elements A in $M(2,k)^n$. Hence $\tilde{x}(g) = 0$ for all $x \in V(R)$. Because of the Nullstellensatz, this implies $g \in \sqrt{R}$, so $g+\sqrt{R} = 0$.

§2. <u>The singular locus of $\mathbb{A}^n * \mathbb{A}^n$ and $Y-Y_S$</u>.

In prop. 1.7. we have shown that $\mathbb{A}^n * \mathbb{A}^n = \operatorname{Spec} B$ where B is the subalgebra of $A = k[X_1,\dots,X_n,Y_1,\dots,Y_n]$ generated by X_i+Y_i $(1 \le i \le n)$ and $(X_i-Y_i)(X_j-Y_j)$ $(1 \le i \le j \le n)$.

<u>definition 2.1</u>. Let $A' = k[U_1,\dots,U_n,V_1,\dots,V_n]$ and $\beta : A' \overset{\sim}{\to} A$ the k-algebra isomorphism such that $\beta(U_i) = X_i-Y_i$ and $\beta(V_i) = X_i+Y_i$ $(1 \le i \le n)$. Let $B' = \beta^{-1}(B)$. Obviously B' is the subalgebra of A' generated by V_i $(1 \le i \le n)$ and U_iU_j $(1 \le i \le j \le n)$.

<u>definition 2.2</u>. Let $\Sigma_0 = k[\dots Z_{ij}\dots]_{1\le i\le j\le n}$. We write $Z_{ij} = Z_{ji}$ if $i \ge j$. Define the k-algebra homomorphism $\theta : \Sigma_0 \to k[U_1,\dots,U_n]$ by $\theta(Z_{ij}) = U_iU_j$.

theorem 2.3. $B \simeq B' \simeq k[V_1,\dots,V_n] \otimes_k {}^{\Sigma_0}/I_0$ where I_0 is the (homogeneous) ideal of Σ_0 generated by the elements $Z_{ij}Z_{kl}-Z_{ik}Z_{jl}$ $(i,j,k,l \in \{1,\dots,n\})$.

proof: Clearly we only have to show that I_0 equals $\ker\theta$.

Let $v : \mathbb{P}^{n-1} \to \mathbb{P}^{\frac{1}{2}n(n+1)-1} = \operatorname{Proj}\Sigma_0$ be defined by

$$v(u_1,\dots,u_n) = (u_1^2,\dots,u_n^2,u_1u_2,\dots,u_iu_j,\dots,u_{n-1}u_n).$$

v is a Veronese mapping [12]. It is well-known that v is an isomorphic embedding of $\mathbb{P}^{n-1}$ in $\mathbb{P}^{\frac{1}{2}n(n+1)-1}$ and that the Veronese variety $v(\mathbb{P}^{n-1})$ is determined by the above ideal I_0. Hence $\ker\theta = \sqrt{I_0}$.

We now apply lemma IV.2.14. in order to prove $I_0 = \sqrt{I_0}$. Well, $\sqrt{I_0}$ is a prime ideal in Σ_0; $\dim(\operatorname{Spec} {}^{\Sigma_0}/\sqrt{I_0}) = \dim(\mathbb{A}^n{*}\mathbb{A}^n)-n = n$ because $\sqrt{I_0} = \ker\theta$ and $B \simeq k[V_1,\dots,V_n] \otimes_k {}^{\Sigma_0}/\ker\theta$. Let y be the element of $V(I_0) \subset k^{\frac{1}{2}n(n+1)}$ such that $\tilde{y}(Z_{11}) = 1$ and $\tilde{y}(Z_{ij}) = 0$ if $(i,j) \neq (1,1)$. Then

$$\operatorname{rank}\left(\frac{\delta(Z_{ij}Z_{kl}-Z_{ik}Z_{jl})}{\delta Z_{pq}}\right)(y) = \operatorname{rank}\left(\frac{\delta(Z_{11}Z_{ij}-Z_{1i}Z_{1j})}{\delta Z_{pq}}\right)(y)_{1<i\le j\le n} = n-1+\binom{n-1}{2} = \binom{n}{2}.$$

Therefore $I_0 = \sqrt{I_0}$.

proposition 2.4. The singular locus of $\mathbb{A}^n{*}\mathbb{A}^n$, and also the one of $Y-Y_S$, is isomorphic with $\mathbb{A}^n$.

proof: Since $v(\mathbb{P}^{n-1})$ is isomorphic to the smooth variety $\mathbb{P}^{n-1}$, 0 is the only singular point of $\operatorname{Spec} {}^{\Sigma_0}/I_0$. Now $\mathbb{A}^n{*}\mathbb{A}^n \simeq \mathbb{A}^n \times_k \operatorname{Spec} {}^{\Sigma_0}/I_0$ (2.3) and therefore, the singular locus of $\mathbb{A}^n{*}\mathbb{A}^n$ is isomorphic with $\mathbb{A}^n$.

remark: If L denotes the singular locus of $Y-Y_S$ and $\phi : X \to Y$ is as usual, it is easy to see that the closed points of $\phi^{-1}(L)$ correspond to the reducible elements $(A_1,\dots,A_n)$ in $M(2,k)^n$ such that each A_i has just one eigenvalue.

Chapter VI. The moduli problem.

§1. A coarse moduli space.

The concepts of the first part of this thesis can be generalized as follows. Suppose we have on a variety S a vector bundle of rank 2. Intuitively this is a family of 2-dimensional vector spaces over k parametrized by the variety S. Suppose also an n-tuple $(A_1,\dots,A_n)$ of endomorphisms of this vector bundle is given, which on each fibre corresponds to an element of $M(2,k)^n$. Is it possible to classify this type of algebraic objects up to isomorphisms? Before answering a question one must know the meaning of the words used in its formulation.

Varieties are as always reduced, separated k-schemes of finite type. The algebraic counterpart of a bundle on a variety $(S,\mathcal{O}_S)$ is a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. The algebraic objects we study are given in:

definition 1.1. An algebraic family of n-tuples of endomorphisms on a variety S, in short a family on S, is an object of the form $(E;A_1,\dots,A_n)$, E being a locally free sheaf of $\mathcal{O}_S$-modules of rank 2 and A_i an endomorphism of E for all $1 \leq i \leq n$. Two such families $(E;A_1,\dots,A_n)$ and $(E';A_1',\dots,A_n')$ are said to be equivalent, denoted by $(E;A_1,\dots,A_n) \sim (E';A_1',\dots,A_n')$, if and only if there exists an isomorphism $T : E' \xrightarrow{\sim} E$ such that $T^{-1}A_iT = A_i'$ for all i. The equivalence class of $(E;A_1,\dots,A_n)$ is denoted by $cl(E;A_1,\dots,A_n)$.

If $f : X \to Y$ is a morphism of varieties and $(E;A_1,\dots,A_n)$ is a family on Y, then $(f^*E;f^*A_1,\dots,f^*A_n)$ is a family on X. Evidently this pull-back of a family on Y respects the equivalence-type of it.

<u>definition 1.2</u>. For all varieties S we define $F(S)$ to be the set of equivalence classes of families on S. Denoting the category of varieties by Var, in this way we obtain a contravariant functor F from Var to Ens (the category of sets).

<u>remark</u>: $F(\mathrm{Spec}\ k) \equiv \{\mathrm{cl}(A_1,\dots,A_n) \mid (A_1,\dots,A_n) \in M(2,k)^n\}$ (cf. II.1.1.)

The question we started with, can now be sharpened into asking: <u>Is it possible to classify the contravariant functor F : Var $\to$ Ens</u>? The only mystification left is the meaning of the word "classify" in this context. The strongest possible meaning of it is given in the following definition.

<u>definition 1.3</u>. The contravariant functor F : Var $\to$ Ens is representable iff there exists a variety M and an isomorphism $\Phi : F \xrightarrow{\sim} h_M$ of contravariant functors, where h_M : Var $\to$ Ens is defined by $h_M(S) = \mathrm{Hom}(S,M)$. In this case the pair (M,Φ) is called a fine moduli space for the algebraic families of n-tuples of endomorphisms.

A weaker form of classifying F is given by:

<u>definition 1.4</u>. A coarse moduli space for the families in question is a pair (M,Φ) consisting of a variety M and a morphism of functors $\Phi : F \to h_M$ fulfilling two conditions:

(1) $\Phi(\mathrm{Spec}\ k) : F(\mathrm{Spec}\ k) \to \mathrm{Hom}(\mathrm{Spec}\ k,M) \equiv M$ is bijective.

(2) The universal property: for each variety N and each morphism of functors $\Psi : F \to h_N$ there is a unique morphism $f : M \to N$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & h_M \arrow[d, "\mathrm{Hom}(\ ,f)"] \\ & h_N \end{tikzcd}$$

is commutative.

Now the ultimate version of our question is: <u>Does there exist a fine or a coarse moduli space for our functor F?</u>

Notice that both types of moduli spaces are unique up to isomorphism, if they exist at all. A fine moduli space is obviously a coarse moduli space, so it is natural to first search for a coarse moduli space. If it exists and F is representable it must be a fine moduli space too.

In I, §1 we defined $X = \text{Spec } R$, $R = k[\underline{X}_1,\ldots,\underline{X}_n]$ and an action of $G = \text{Spec } S$ on X. On the variety X we have a standard family $(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)$ where $\underline{X}_i$ now represents the endomorphism of $\mathcal{O}_X^2$ with matrix $\begin{pmatrix} X_{i;11} & X_{i;12} \\ X_{i;21} & X_{i;22} \end{pmatrix} \in M(2,\Gamma(X,\mathcal{O}_X)) = M(2,R)$.

Hence we have a standard element of $F(X)$ namely $\text{cl}(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)$.

Now suppose M is a variety and $\Phi : F \to h_M$ is a morphism of functors. Then $\text{cl}(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n) \in F(X)$ gives us a morphism $\phi : X \to M$ defined by $\phi = \Phi(X)[\text{cl}(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)]$.

<u>proposition 1.5</u>. This $\phi : X \to M$ is constant on the orbits of closed points of X under the action of G on X.

<u>proof</u>: Let $x \in X$ be a closed point corresponding to $(A_1,\ldots,A_n) \in M(2,k)^n$ where $A_i = \begin{pmatrix} a_{i;11} & a_{i;12} \\ a_{i;21} & a_{i;22} \end{pmatrix}$. The element $(A_1,\ldots,A_n)$ determines a k-algebra homomorphism $\alpha^* : R \to k$ with $\alpha^*(X_{i;pq}) = a_{i;pq}$ and α^* induces a morphism $\alpha : \text{Spec } k \to X$.

$F(\alpha) : F(X) \to F(\text{Spec } k)$ and $F(\alpha)[\text{cl}(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] = \text{cl}(k^2;A_1,\ldots,A_n)$.

Now suppose x' is a closed point in the orbit of x. Say x' corresponds to $(T^{-1}A_1T,\ldots,T^{-1}A_nT)$ with $T \in Gl(2,k)$.

$(T^{-1}A_1T,\ldots,T^{-1}A_nT)$ gives rise to a morphism $\beta : \text{Spec } k \to X$. We have:

$F(\beta)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] = cl(k^2;T^{-1}A_1T,\ldots,T^{-1}A_nT) = cl(k^2;A_1,\ldots,A_n)$.

Now, in view of the commutativity of the adjacent diagrams - corresponding to α and β respectively - we have:

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi(X)} & \mathrm{Hom}(X,M) \\ F(\alpha)\Big\downarrow\Big\downarrow F(\beta) & & h_M(\alpha)\Big\downarrow\Big\downarrow h_M(\beta) \\ F(\mathrm{Spec}\ k) & \xrightarrow[\Phi(\mathrm{Spec}\ k)]{} & \mathrm{Hom}(\mathrm{Spec}\ k,M) \end{array}$$

$$\begin{aligned}\phi(x) &= h_M(\alpha)\Phi(X)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] \\ &= \Phi(\mathrm{Spec}\ k)F(\alpha)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] \\ &= \Phi(\mathrm{Spec}\ k)[cl(k^2;A_1,\ldots,A_n)] \\ &= \Phi(\mathrm{Spec}\ k)F(\beta)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] \\ &= h_M(\beta)\Phi(X)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] \\ &= \phi(x').\end{aligned}$$

<u>definition 1.6</u>. Let C denote the set of morphisms from X to M being constant on the orbits of closed points of X. Let $\mathrm{Hom}(F,h_M)$ be the set of morphisms of functors $F \to h_M$. Define the map $\nabla : \mathrm{Hom}(F,h_M) \to C$ by $\nabla(\Phi) = \Phi(X)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)]$.

<u>proposition 1.7</u>. The map ∇ is bijective.

<u>proof</u>: (1) Let $\Phi,\Phi' \in \mathrm{Hom}(F,h_M)$ such that $\nabla\Phi = \nabla\Phi'$, i.e.,

$$\phi = \Phi(X)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)] = \Phi'(X)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)]$$

Let S be a variety and $cl(E;A_1,\ldots,A_n) \in F(S)$.

Say $f = \Phi(S)[cl(E;A_1,\ldots,A_n)]$ and $f' = \Phi'(S)[cl(E;A_1,\ldots,A_n)]$.

We have to show that $f = f' \in \mathrm{Hom}(S,M)$.

Since E is a locally free sheaf of $\mathcal{O}_S$-modules of rank 2, there is an affine open covering $\{U_\alpha\}_{\alpha\in I}$ of S such that $E|U_\alpha \stackrel{\sim}{=} \mathcal{O}_{U_\alpha}^2$. Choosing a basis in $E|U_\alpha$ we can identify $A_i|U_\alpha$ with a matrix $\begin{pmatrix} a_{i;11}^\alpha & a_{i;12}^\alpha \\ a_{i;21}^\alpha & a_{i;22}^\alpha \end{pmatrix}$ where $a_{i;pq}^\alpha \in \Gamma(U_\alpha,\mathcal{O}_S)$.

Because $\mathrm{Hom}(U_\alpha, X) \simeq \mathrm{Hom}_{k\text{-alg}}(R, \Gamma(U_\alpha, \mathcal{O}_S))$ we get a morphism $e_\alpha : U_\alpha \to X$ determined by the k-algebra homomorphism $\tilde{e}_\alpha : R \to \Gamma(U_\alpha, \mathcal{O}_S)$ such that $\tilde{e}_\alpha(X_{i;pq}) = a^\alpha_{i;pq}$. From this definition it follows that $(E|U_\alpha; A_1|U_\alpha, \ldots, A_n|U_\alpha)$ is equivalent to $(e^*_\alpha \mathcal{O}^2_X; e^*_\alpha \underline{X}_1, \ldots, e^*_\alpha \underline{X}_n)$ for all $\alpha \in I$. Now $f = f'$ if and only if $f|U_\alpha = f'|U_\alpha$ for all $\alpha \in I$. Notice that the following diagram is commutative.

$$\begin{tikzcd}
F(U_\alpha) \arrow[r, "\Phi(U_\alpha)"] & \mathrm{Hom}(U_\alpha, M) \\
F(X) \arrow[u, "F(e_\alpha)"] \arrow[r, "\Phi(X)"] & \mathrm{Hom}(X, M) \arrow[u, "h_M(e_\alpha)"']
\end{tikzcd}$$

Therefore we have:

$$\begin{aligned}
\phi \circ e_\alpha = h_M(e_\alpha)\phi &= h_M(e_\alpha)\Phi(X)[\mathrm{cl}(\mathcal{O}^2_X; \underline{X}_1, \ldots, \underline{X}_n)] \\
&= \Phi(U_\alpha)F(e_\alpha)[\mathrm{cl}(\mathcal{O}^2_X; \underline{X}_1, \ldots, \underline{X}_n)] \\
&= \Phi(U_\alpha)[\mathrm{cl}(e^*_\alpha \mathcal{O}^2_X; e^*_\alpha \underline{X}_1, \ldots, e^*_\alpha \underline{X}_n)] \\
&= \Phi(U_\alpha)[\mathrm{cl}(E|U_\alpha; A_1|U_\alpha, \ldots, A_n|U_\alpha)] \\
&= f|U_\alpha.
\end{aligned}$$

Analogously: $\phi e_\alpha = f'|U_\alpha$. Hence $f|U_\alpha = f'|U_\alpha$ for all $\alpha \in I$.

(2) Let $\phi \in C$. We shall define a morphism of functors $\Phi : F \to h_M$ such that $\nabla(\Phi) = \phi$.

Let S be a variety and $\mathrm{cl}(E; A_1, \ldots, A_n) \in F(S)$. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of S such that $E|U_\alpha \simeq \mathcal{O}^2_{U_\alpha}$. Choose a basis in $E|U_\alpha$ and construct a morphism $e_\alpha : U_\alpha \to X$ just as in (1). Define $g_\alpha : U_\alpha \to M$ by $g_\alpha = \phi \circ e_\alpha$. Since ϕ is constant on the orbits of closed points of X and since e_α only depends on the choice of a basis in $E|U_\alpha$, it is clear that $\phi e_\alpha(x) = \phi e_\beta(x)$ for all closed points x in $U_\alpha \cap U_\beta$ $(\alpha, \beta \in I)$.

Therefore there exists $g \in \mathrm{Hom}(S,M)$ such that $g|U_\alpha = g_\alpha$ for all $\alpha \in I$. Define $\Phi(S)[\mathrm{cl}(E;A_1,\ldots,A_n)] = g$. Notice that g is independent of the various choices we made. Hence $\Phi(S) : F(S) \to h_M(S)$ is a well-defined map for each variety S.

Next we show that $\Phi \in \mathrm{Hom}(F,h_M)$. Let $\tau : T \to S$ be a morphism of varieties. Then we have to prove the commutativity of the diagram

$$\begin{tikzcd} F(S) \arrow[r, "\Phi(S)"] \arrow[d, "F(\tau)"'] & \mathrm{Hom}(S,M) \arrow[d, "h_M(\tau)"] \\ F(T) \arrow[r, "\Phi(T)"] & \mathrm{Hom}(T,M) \end{tikzcd}$$

Let $\xi = \mathrm{cl}(E;A_1,\ldots,A_n) \in F(S)$. Put $g = \Phi(S)\xi : S \to M$. Let $\{U_\alpha\}_{\alpha\in I}$ be an open covering of S such that $E|U_\alpha \simeq \mathcal{O}^2_{U_\alpha}$. Then $\{\tau^{-1}U_\alpha\}_{\alpha\in I}$ is an open covering of T such that $\tau^*E|\tau^{-1}U_\alpha \simeq \mathcal{O}^2_{\tau^{-1}U_\alpha}$. Since h_M is a sheaf it is sufficient to prove that $[\Phi(T)F(\tau)\xi]|\tau^{-1}U_\alpha = [h_M(\tau)\Phi(S)\xi]|\tau^{-1}U_\alpha$ for all $\alpha \in I$.

Choose a basis $(\lambda_1^\alpha,\lambda_2^\alpha)$ of $E|U_\alpha$ which enables us to identify $A_i|U_\alpha$ with a matrix $(a^\alpha_{i;pq})_{p,q} \in M(2,\Gamma(U_\alpha,\mathcal{O}_S))$. Let $e_\alpha : U_\alpha \to X$ be the morphism determined by the k-algebra homomorphism $\tilde{e}_\alpha : R \to \Gamma(U_\alpha,\mathcal{O}_S)$ satisfying $\tilde{e}_\alpha(X_{i;pq}) = a^\alpha_{i;pq}$. Then $g|U_\alpha = \phi \circ e_\alpha$. Hence the following diagram is commutative (defining τ')

$$\begin{tikzcd} \tau^{-1}U_\alpha \arrow[r, hook] \arrow[drr, "\tau'"] & T \arrow[r, "\tau"] & S \arrow[r, "g"] & M \\ & & U_\alpha \arrow[u, hook] \arrow[r, "e_\alpha"'] & X \arrow[u, "\phi"'] \end{tikzcd}$$

Therefore $[h_M(\tau)\Phi(S)\xi]|\tau^{-1}U_\alpha = [g \circ \tau]|\tau^{-1}U_\alpha = \phi \circ e_\alpha \circ \tau'$.

Now choose $(\tau^*\lambda_1^\alpha,\tau^*\lambda_2^\alpha)$ as a basis of $\tau^*E|\tau^{-1}U_\alpha$; identify $\tau^*A_i|\tau^{-1}U_\alpha$

with a matrix $(b^{\alpha}_{i;pq})_{p,q} \in M(2,\Gamma(\tau^{-1}U_\alpha,\mathcal{O}_T))$ and define $\varepsilon_\alpha : \tau^{-1}U_\alpha \to X$ by means of the corresponding k-algebra homomorphism $\tilde{\varepsilon}_\alpha : R \to \Gamma(\tau^{-1}U_\alpha,\mathcal{O}_T)$ such that $\tilde{\varepsilon}_\alpha(X_{i;pq}) = b^{\alpha}_{i;pq}$.
Since $[\Phi(T)F(\tau)\xi]|\tau^{-1}U_\alpha = [\Phi(T)\mathrm{cl}(\tau^*E;\tau^*A_1,\dots,\tau^*A_n)]|\tau^{-1}U_\alpha = \phi \circ \varepsilon_\alpha$ and $\varepsilon_\alpha = \varepsilon_\alpha \circ \tau'$, $\Phi(T)F(\tau)\xi$ and $h_M(\tau)\Phi(S)\xi$ coincide on $\tau^{-1}U_\alpha$.

Finally $\Phi(X)\mathrm{cl}(\mathcal{O}^2_X;\underline{X}_1,\dots,\underline{X}_n) = \phi$ in view of the definition of $\Phi(X)$.

Now suppose (M,Φ) is a coarse moduli space for F. Let $\phi = \nabla(\Phi)$. The universal property can be translated, by means of the above proposition, into: for each variety N and morphism $\psi : X \to N$, constant on the orbits of closed points, there exists a unique morphism $f : M \to N$ such that $\psi = f \circ \phi$. Looking back at the definition of the quotient of X by G (cf. I.2.3.) this implies:

<u>proposition 1.8</u>. Let M be a variety and $\Phi : F \to h_M$ a morphism of functors. Then (M,Φ) is a coarse moduli space iff $\Phi(\mathrm{Spec}\ k)$ is bijective and $(M,\nabla\Phi)$ is a quotient of X by G.

But quotients of X by G are unique up to isomorphism. We already have a quotient, namely (Y,ϕ) where $Y = \mathrm{Spec}\ R^G$ and $\phi : X \to Y$ is induced by $R^G \hookrightarrow R$. Let $\Phi : F \to h_Y$ be the morphism of functors satisfying $\Phi(X)[\mathrm{cl}(\mathcal{O}^2_X;\underline{X}_1,\dots,\underline{X}_n)] = \phi$. Then we can conclude: if there exists a coarse moduli space for F, (Y,Φ) is a coarse moduli space too. Now, does it happen that (Y,Φ) is a coarse moduli space? Well, the only thing left to verify is the bijectivity of $\Phi(\mathrm{Spec}\ k)$. Translated in terms of $\phi : X \to Y$ this means that for each closed point $y \in Y$, $\phi^{-1}(y)$ has to be an orbit. Unfortunately this is not the case, as we already have seen in the introduction of this thesis. So our functor F has no coarse moduli space, let alone a fine moduli space.

Inspired by the way we obtained this totally negative result, we hope to get more success if we restrict the class of objects, i.e., if we look at a subfunctor of F defined in a suitable way. First we give some general observations and definitions.

(i) Let $(S,\mathcal{O}_S)$ be a variety, U an open subset of S and $f \in \Gamma(U,\mathcal{O}_S)$. For each closed point $x \in U$ the image of f under

$$\Gamma(U,\mathcal{O}_S) \to \mathcal{O}_{S,x} \to \mathcal{O}_{S,x}/\underline{m}_x = \kappa(x) \xrightarrow{\sim} k$$

will be denoted by $f(x)$ (the value of f at x).

(ii) Let E be a locally free sheaf of $\mathcal{O}_S$-modules of rank 2, $A \in \mathrm{End}\, E$ and x a closed point of S. Then there is an open neighbourhood U of x such that $E|U \simeq \mathcal{O}_U^2$. We can represent $A|U$ by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with entries in $\Gamma(U,\mathcal{O}_S)$ (depending on the choice of a basis in $E|U$). We define $A(x) \in M(2,k)^2$ to be $\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$. We can cover S by open sets $\{U_\alpha\}_{\alpha \in I}$ such that $E|U_\alpha \simeq \mathcal{O}_{U_\alpha}^2$. Choosing a basis in each $E|U_\alpha$ we can identify $A|U_\alpha$ with a matrix $\begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}$ where $a_\alpha,b_\alpha,c_\alpha,d_\alpha \in \Gamma(U_\alpha,\mathcal{O}_S)$. Since $a_\alpha+d_\alpha$ is independent of the choice of this basis, there exists an element, denoted $\mathrm{Tr}A$, in $\Gamma(S,\mathcal{O}_S)$ such that $(\mathrm{Tr}A)|U_\alpha = a_\alpha+d_\alpha$ for all $\alpha \in I$. In this way we get a map $\mathrm{Tr} : \mathrm{End}\, E \to \Gamma(S,\mathcal{O}_S)$.

<u>definition 1.9</u>. Let E be a locally free sheaf of $\mathcal{O}_S$-modules of rank 2 on a variety S. Let $A_1,\ldots,A_n \in \mathrm{End}\, E$. Then $(A_1,\ldots,A_n)$ is said to be irreducible iff $(A_1(x),\ldots,A_n(x)) \in M(2,k)^n$ is irreducible for all closed points $x \in S$.

<u>remark 1.10</u>. In view of II, §1 we have:

(A_1,A_2) irreducible $\Leftrightarrow$ $[\mathrm{Tr}A_1^2A_2^2-\mathrm{Tr}(A_1A_2)^2](x) \neq 0$ for all closed points $x \in S$. If $n \geq 3$: $(A_1,\ldots,A_n)$ irreducible $\Leftrightarrow$ for all closed points $x \in S$ there exist i,j and k with $1 \leq i<j<k \leq n$ such that $(A_i(x),A_j(x),A_k(x))$ is

irreducible $\Leftrightarrow$ for all closed points $x \in S$ there exist $1 \leq i < j < k \leq n$ such that the value at x of at least one of the elements $TrA_i^2A_j^2 - Tr(A_iA_j)^2$, $TrA_i^2A_k^2 - Tr(A_iA_k)^2$, $TrA_j^2A_k^2 - Tr(A_jA_k)^2$ and $TrA_iA_jA_k - TrA_iA_kA_j$ of $\Gamma(S, \mathcal{O}_S)$ is different from zero.

<u>definition 1.11</u>. Let $G : Var \to Ens$ be the subfunctor of F such that $G(S) = \{cl(E;A_1,\ldots,A_n) | (A_1,\ldots,A_n) \text{ irreducible}\}$.

Notice that G indeed is a subfunctor of F. This follows from the above remark and the fact that, if $f : S \to T$ is a morphism of varieties, E is a locally free sheaf of $\mathcal{O}_T$-modules of rank 2 and $A \in End\ E$, $(Trf^*A)(x) = (TrA)(f(x))$ for all closed points x of S.

From the first part of this thesis and the discussion above it follows that we expect Y_S to be a coarse moduli space for G. Well let us prove it.

We had $cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n) \in F(X)$. The morphism $\phi : X \to Y$ determines a morphism of functors $\Phi : F \to h_Y$ such that $\phi = \Phi(X)[cl(\mathcal{O}_X^2;\underline{X}_1,\ldots,\underline{X}_n)]$. Explicitly: if S is a variety and $cl(E;A_1,\ldots,A_n) \in F(S)$, then $\Phi(S)[cl(E;A_1,\ldots,A_n)]$ is the morphism from S to Y determined by the k-algebra homomorphism $\Sigma/J \overset{\sim}{\to} R^G \to \Gamma(S,\mathcal{O}_S)$ where

$$\begin{cases} X_i + J \mapsto TrA_i & (1 \leq i \leq n) \\ Y_i + J \mapsto 2TrA_i^2 - (TrA_i)^2 & (1 \leq i \leq n) \\ Z_{ij} + J \mapsto 2TrA_iA_j - TrA_iTrA_j & (1 \leq i < j \leq n) \\ W_{ijk} + J \mapsto TrA_iA_jA_k - TrA_iA_kA_j & (1 \leq i < j < k \leq n) \end{cases}$$

If $cl(E;A_1,\ldots,A_n) \in G(S)$, $(A_1,\ldots,A_n)$ is irreducible and therefore the morphism $\Phi(S)[cl(E;A_1,\ldots,A_n)]$ maps S into Y_S. So we have a morphism of functors $\Phi' : G \to h_{Y_S}$ such that the diagram below commutes.

Let $\xi = \mathrm{cl}(\mathcal{O}^2_{X_S};\underline{X}_1|X_S,\ldots,\underline{X}_n|X_S)$. Then $\xi \in G(X_S)$ and $\Phi'(X_S)(\xi) = \phi' : X_S \to Y_S$.

$$\begin{tikzcd} F \arrow[r, "\Phi"] & h_Y \\ G \arrow[u, hook] \arrow[r, "\Phi'"] & h_{Y_S} \arrow[u, hook] \end{tikzcd}$$

If M is a variety and $\Psi : G \to h_M$ is a morphism of functors, again it is easy to see that $\Psi(X_S)(\xi) : X_S \to M$ is constant on the orbits of closed points of X_S under the action of G on X_S (cf. 1.5.). Since (Y_S,ϕ') is a quotient of X_S by G, it follows that (Y_S,Φ') has the universal property (cf.1.4.). But moreover, we know that (Y_S,ϕ') is a geometric quotient of X_S by G (cf. I, §2). This especially implies that $(\phi')^{-1}(y)$ is an orbit for each closed point $y \in Y_S$, from which, in turn, it follows that $\Phi'(\mathrm{Spec}\ k) : G(\mathrm{Spec}\ k) \to \mathrm{Hom}(\mathrm{Spec}\ k,Y_S)$ is bijective. So we have proved:

theorem 1.12. (Y_S,Φ') is a coarse moduli space for the functor G.

It is natural to ask whether (Y_S,Φ') is a fine moduli space for G, i.e., whether $\Phi' : G \to h_{Y_S}$ is an isomorphism of functors. The answer is no. Take for instance a variety S with a non-trivial invertible $\mathcal{O}_S$-module L on it. Then $\Phi'(S)[\mathrm{cl}(E;A_1,\ldots,A_n)]=\Phi'(S)[\mathrm{cl}(E\otimes L;A_1\otimes 1,\ldots,A_n\otimes 1)]$. Hence $\Phi'(S)$ is not injective for $E\otimes L$ is certainly not isomorphic to E.

So the best result for G is a coarse moduli space.

§2. The case n = 2; the introduction of roots; the sheafification of a functor.

In order to understand what keeps G off from being representable, we indicate the results attained by Dekkers in the case n = 2 (cf. [2]).

$Y = \mathbb{A}^5 = \mathrm{Spec}\ k[X_1,X_2,Y_1,Y_2,Z_{12}]$; $Y_S = \mathbb{A}^5 - V(Z_{12}^2 - Y_1Y_2)$.

G : Var $\to$ Ens such that $G(S) = \{cl(E;A_1,A_2) | (A_1,A_2)$ irreducible$\}$.

$\Phi' : G \to h_{Y_S}$ is the morphism of functors such that, for each variety S and each $cl(E;A_1,A_2) \in G(S)$, $\Phi'(S)[cl(E;A_1,A_2)]$ is the morphism $S \to Y_S$ assigning to a closed point $x \in S$:

$(TrA_1(x), TrA_2(x), [2TrA_1^2-(TrA_1)^2](x), [2TrA_2^2-(TrA_2)^2](x), [2TrA_1A_2-TrA_1TrA_2](x))$.

A necessary (and sufficient) condition for Φ' to be an epimorphism of functors is the existence of an element $\eta \in G(Y_S)$ such that $\Phi'(Y_S)(\eta) = id_{Y_S}$. Dekkers has shown that η does not exist. But he showed more: in proving that (Y_S,Φ') is a coarse moduli space he defines a twofold covering M of Y_S as follows: M is the subvariety of $\mathbb{A}^6 = \operatorname{Spec} k[X_1,X_2,Y_1,Y_2,Z_{12},R]$ defined by $M = V(R^2-(Z_{12}^2-Y_1Y_2))-V(R)$.
The canonical projection $\operatorname{Spec} k[X_1,X_2,Y_1,Y_2,Z_{12},R] \to \operatorname{Spec} k[X_1,X_2,Y_1,Y_2,Z_{12}]$ induces the projection $p : M \to Y_S$.

<u>proposition 2.1</u>. There exists $\xi_0 \in G(M)$ such that $\Phi'(M)(\xi_0) = p$ (cf. [2]).

<u>proof</u>: For convenience of the reader we repeat Dekker's proof using the notations of this thesis.

Let $U_1 = V(R^2-(Z_{12}^2-Y_1Y_2))-V(Z_{12}-R) \cap V(R)$ and

$U_2 = V(R^2-(Z_{12}^2-Y_1Y_2))-V(Z_{12}+R) \cap V(R)$.

Then U_1 and U_2 are open subsets of M and $M = U_1 \cup U_2$. Define two elements of $M(2,\Gamma(U_1,\mathcal{O}_M))$ as follows:

$$A_1^1 = \tfrac{1}{2}\begin{pmatrix} X_1 & 1 \\ Y_1 & X_1 \end{pmatrix} \text{ and } A_2^1 = \tfrac{1}{2}\begin{pmatrix} X_2 & Y_2(Z_{12}-R)^{-1} \\ Z_{12}-R & X_2 \end{pmatrix}.$$

And also two elements of $M(2,\Gamma(U_2,\mathcal{O}_M))$:

$$A_1^2 = \tfrac{1}{2}\begin{pmatrix} X_1 & Y_1 \\ 1 & X_1 \end{pmatrix} \text{ and } A_2^2 = \tfrac{1}{2}\begin{pmatrix} X_2 & Z_{12}+R \\ Y_2(Z_{12}+R)^{-1} & X_2 \end{pmatrix}.$$

Define $T \in Gl(2,\Gamma(U_1 \cap U_2,\mathcal{O}_M))$ to be $T = \begin{pmatrix} Y_1 & 0 \\ 0 & 1 \end{pmatrix}$.

The pair (A_1^i,A_2^i) represents an irreducible pair of endomorphisms of $\mathcal{O}_{U_i}^2$ $(i = 1,2)$.

Using T as a morphism of transition we construct a sheaf $\mathcal{E}_0$ of $\mathcal{O}_M$-modules, which is obviously the sum of $\mathcal{O}_M$ and a line bundle, such that $\mathcal{E}_0$ is free of rank 2 on U_1 and U_2 (cf. [7]). Now on $U_1 \cap U_2$ we have $(TA_1^1T^{-1},TA_2^1T^{-1}) = (A_1^2,A_2^2)$, so the pairs (A_1^1,A_2^1) and (A_1^2,A_2^2) induce an irreducible pair of endomorphisms of $\mathcal{E}_0$, say (A_1,A_2). Defining $\xi_0 = \mathrm{cl}(\mathcal{E}_0;A_1,A_2)$ one easily verifies $\Phi'(M)\xi_0 = p$.

Now notice that it was possible to define $\xi_0 \in \mathcal{G}(M)$ because of the existence of a regular function $R \in \Gamma(M,\mathcal{O}_M)$ such that $R^2 = Z_{12}^2-Y_1Y_2 = 4\mathrm{Tr}(A_1A_2)^2-4\mathrm{Tr}A_1^2A_2^2$ (cf. III.1.3.) If we "include" in our functor such a root we may possibly get a representable one.

<u>definition 2.2</u>. Let S be a variety. Then $\mathcal{P}(S)$ denotes the set of equivalence classes of objects of the form $(\mathcal{E};A_1,A_2;r)$ where

(1) $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_S$-modules of rank 2,

(2) A_1 and A_2 are endomorphisms of $\mathcal{E}$, and

(3) $r \in \Gamma(S,\mathcal{O}_S)$ such that $r^2 = 4\mathrm{Tr}(A_1A_2)^2-4\mathrm{Tr}A_1^2A_2^2$ and $r(x) \neq 0$ for all closed points $x \in S$, or equivalently: (A_1,A_2) is irreducible.

$$\begin{aligned}(\mathcal{E};A_1,A_2;r) \sim (\mathcal{E}';A_1',A_2';r') &\Leftrightarrow (\mathcal{E};A_1,A_2) \sim (\mathcal{E}';A_1',A_2') \text{ and } r = r' \\ &\Leftrightarrow r = r' \text{ and there exists an isomorphism} \\ &\qquad T : \mathcal{E}' \xrightarrow{\sim} \mathcal{E} \text{ such that } (A_1',A_2')=(T^{-1}A_1T,T^{-1}A_2T).\end{aligned}$$

The equivalence class of $(\mathcal{E};A_1,A_2;r)$ is denoted by $\mathrm{cl}(\mathcal{E};A_1,A_2;r)$.

If $f : S \to T$ is a morphism of varieties and $\mathrm{cl}(\mathcal{E};A_1,A_2;r) \in \mathcal{P}(T)$ it is clear that $\mathrm{cl}(f^*\mathcal{E};f^*A_1,f^*A_2;f^*r)$ belongs to $\mathcal{P}(S)$. In this way we get a functor $\mathcal{P} : \mathrm{Var} \to \mathrm{Ens}$.

The morphism $\Phi' : \mathcal{G} \to h_{Y_S}$ can be extended as follows:

<u>definition 2.3</u>. $\Phi : \mathcal{P} \to h_M$ is the morphism of functors such that $\Phi(S)[\mathrm{cl}(\mathcal{E};A_1,A_2;r)]$ is the element of $\mathrm{Hom}(S,M)$ deduced from the k-algebra homomorphism $k[X_1,X_2,Y_1,Y_2,Z_{12},R] \to \Gamma(S,\mathcal{O}_S)$ satisfying:

$$\begin{aligned} X_i &\mapsto \mathrm{Tr}A_i \\ Y_i &\mapsto 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 \\ Z_{12} &\mapsto 2\mathrm{Tr}A_1A_2-\mathrm{Tr}A_1\mathrm{Tr}A_2 \\ R &\mapsto r \end{aligned}$$

If we include the element R of $\Gamma(M,\mathcal{O}_M)$ in $\xi_0 \in \mathcal{G}(M)$, we get $\xi = \mathrm{cl}(\mathcal{E}_0;A_1,A_2;R) \in \mathcal{P}(M)$ and obviously we have $\Phi(M)\xi = \mathrm{id}_M$.

<u>proposition 2.4</u>. $\Phi : \mathcal{P} \to h_M$ is an epimorphism.

<u>proof</u>: Let S be a variety and $\sigma \in \mathrm{Hom}(S,M)$. Then we have the commutative diagram:

$$\begin{tikzcd} \mathcal{P}(S) \arrow[r, "\Phi(S)"'] & \mathrm{Hom}(S,M) \\ \mathcal{P}(M) \arrow[u, "\mathcal{P}(\sigma)"] \arrow[r, "\Phi(M)"'] & \mathrm{Hom}(M,M) \arrow[u, "\sigma^*"'] \end{tikzcd}$$

Hence $\Phi(S)\mathcal{P}(\sigma)\xi = \sigma^*\Phi(M)\xi = \sigma^*(\mathrm{id}_M) = \sigma$ and therefore $\Phi(S)$ is surjective.

However on the basis of the same argument as in the case of $\Phi' : \mathcal{G} \to h_{Y_S}$, we see that $\Phi : \mathcal{P} \to h_M$ is still not a monomorphism. Obviously the requirement of the existence of a <u>global</u> isomorphism $T : \mathcal{E}' \overset{\sim}{\to} \mathcal{E}$ in our definition of equivalence is the fundamental obstruction. This presumption is even strengthened by the following theorem, due to Dekkers (given here without proof; cf. [2]):

<u>theorem 2.5</u>. If $\xi,\xi' \in \mathcal{P}(S)$ and $\Phi(S)\xi = \Phi(S)\xi' \in \mathrm{Hom}(S,M)$, then there exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of S such that $\xi|U_\alpha = \xi'|U_\alpha$ in $\mathcal{P}(U_\alpha)$ for all $\alpha \in I$.

We now use [1] in order to construct a new functor P^a, deduced from P in a canonical way, which will appear to be representable.

Let X be a variety and h_X : Var → Ens where $h_X(S)$ = Hom(S,M). Then the functor h_X has the following well-known properties:

(a) Given a variety S, an open covering $\{U_\alpha\}_{\alpha\in I}$ of S and two elements $s,t \in \mathrm{Hom}(S,X)$ such that $s|U_\alpha = t|U_\alpha$ for all $\alpha \in I$, then $s = t$.

(b) Given a variety S, an open covering $\{U_\alpha\}_{\alpha\in I}$ of S and for each $\alpha \in I$ an element $s_\alpha \in \mathrm{Hom}(U_\alpha,X)$ such that $s_\alpha|U_\alpha \cap U_\beta = s_\beta|U_\alpha \cap U_\beta$ for all $\alpha,\beta \in I$, then there exists a (unique) morphism $s \in \mathrm{Hom}(S,X)$ with $s|U_\alpha = s_\alpha$ for all $\alpha \in I$.

In other words: if we restrict the functor h_X to the category of the open subsets of a variety S we get a sheaf of sets on S.

<u>definition 2.6</u>. A contravariant functor F : Var → Ens is called a sheaf iff the restriction of F to the category of the open subsets of a variety S is a sheaf of sets for all S.

<u>remark</u>: In this sense h_X is a sheaf. So, in order to be representable, a functor must necessarily be a sheaf.

<u>definition 2.7</u>. <u>The sheafification of a contravariant functor</u>.

Let F : Var → Ens be a contravariant functor. Then there exists a sheaf F^a : Var → Ens and a morphism of functors $\omega : F \to F^a$ satisfying the universal property:

For each sheaf G : Var → Ens and each morphism of functors $\Psi : F \to G$ there exists a unique morphism $\Psi^a : F^a \to G$ such that $\Psi^a \circ \omega = \Psi$.

F^a is called the associated sheaf of F.

When $\xi \in F(U)$, $x \in U$, we denote by ξ_x the germ of ξ at x with respect to the Zariski topology.

Explicitly:

(1) An element of $\mathcal{F}^a(S)$ can be represented by $\{(U_\alpha,\xi^\alpha)\}_{\alpha\in I}$ where $\{U_\alpha\}_\alpha$ is a Zariski-open covering of S, $\xi^\alpha \in \mathcal{F}(U_\alpha)$ for all $\alpha \in I$ and

$$\forall\alpha,\beta \in I \ \forall x \in U_\alpha \cap U_\beta[\xi_x^\alpha = \xi_x^\beta].$$

(2) $\{(U_\alpha,\xi^\alpha)\}_{\alpha\in I}$ and $\{(V_\beta,\eta^\beta)\}_{\beta\in J}$ represent the same element of $\mathcal{F}^a(S)$ iff $\forall\alpha \in I,\ \beta \in J\ \forall x \in U_\alpha \cap U_\beta[\xi_x^\alpha = \eta_x^\beta]$.

(3) If $\xi \in \mathcal{F}(S)$ then $\omega(\xi)$ is the element of $\mathcal{F}^a(S)$ represented by (S,ξ).

(4) $\Psi : \mathcal{F} \to \mathcal{G}$ as above; $\xi \in \mathcal{F}^a(S)$ represented by $\{(U_\alpha,\xi^\alpha)\}_{\alpha\in I}$. Then $\Psi(U_\alpha)\xi^\alpha \in \mathcal{G}(U_\alpha)$ for all $\alpha \in I$. Since $\xi_x^\alpha = \xi_x^\beta$ for all $x \in U_\alpha \cap U_\beta$ we also have $(\Psi(U_\alpha)\xi^\alpha)_x = (\Psi(U_\beta)\xi^\beta)_x$ if $x \in U_\alpha \cap U_\beta$. However $\mathcal{G}$ is a sheaf and therefore there is $\eta \in \mathcal{G}(S)$ such that $\eta|U_\alpha = \Psi(U_\alpha)\xi^\alpha$ for all $\alpha \in I$. Now $\Psi^a(S)\xi$ is defined to be this η.

Returning to the situation of definition 2.3 and using the fact that h_M is a sheaf we see that there exists a unique morphism of functors Φ^a such that we get a commutative diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\Phi"] \arrow[d, "\omega"'] & h_M \\ \mathcal{P}^a \arrow[ur, "\Phi^a"'] & \end{tikzcd}$$

<u>proposition 2.8</u>. $\Phi^a : \mathcal{P}^a \to h_M$ is an isomorphism of functors. In other words: (M,Φ^a) is a fine moduli space for the functor $\mathcal{P}^a$: Var $\to$ Ens.

<u>proof</u>: We already know that $\Phi : \mathcal{P} \to h_M$ is an epimorphism (2.4.) Because of the commutativity of the diagram above, Φ^a is also an epimorphism. The injectivity of $\Phi^a(S) : \mathcal{P}^a(S) \to \mathrm{Hom}(S,M)$ for an arbitrary variety S has to be proved yet.

Let $\eta, \eta' \in P^a(S)$ be represented by $\{(U_\alpha, \eta^\alpha)\}_{\alpha \in I}$ and $\{(V_\beta, {}'\eta^\beta)\}_{\beta \in J}$ respectively and suppose $\Phi^a(S)\eta = \Phi^a(S)\eta'$. We claim $\eta = \eta'$, i.e.,
$\forall \alpha \in I,\ \beta \in J\ \forall x \in U_\alpha \cap V_\beta\ [\eta^\alpha_x = {}'\eta^\beta_x]$.
$\eta^\alpha \in P(U_\alpha) \Rightarrow \eta^\alpha | U_\alpha \cap V_\beta \in P(U_\alpha \cap V_\beta)$. Now the following diagram commutes:

$$\begin{tikzcd}
P^a(S) \arrow[r, "\Phi^a(S)"] \arrow[d] & \mathrm{Hom}(S,M) \arrow[d] \\
P^\alpha(U_\alpha \cap V_\beta) \arrow[r, "\Phi^a(U_\alpha \cap V_\beta)"] & \mathrm{Hom}(U_\alpha \cap V_\beta, M) \\
P(U_\alpha \cap V_\beta) \arrow[u, "\omega(U_\alpha \cap V_\beta)"] \arrow[ur, "\Phi(U_\alpha \cap V_\beta)"'] &
\end{tikzcd}$$

Hence
$$\begin{aligned}
[\Phi^a(S)\eta]|_{U_\alpha \cap V_\beta} &= \Phi^a(U_\alpha \cap V_\beta)(\eta | U_\alpha \cap V_\beta) \\
&= \Phi^a(U_\alpha \cap V_\beta)\omega(U_\alpha \cap V_\beta)(\eta^\alpha | U_\alpha \cap V_\beta) \\
&= \Phi(U_\alpha \cap V_\beta)(\eta^\alpha | U_\alpha \cap V_\beta)
\end{aligned}$$

Similarly we have $[\Phi^a(S)\eta']\big|_{U_\alpha \cap V_\beta} = \Phi(U_\alpha \cap V_\beta)({}'\eta^\beta | U_\alpha \cap V_\beta)$.
Since $\Phi^a(S)\eta = \Phi^a(S)\eta'$ we now have two elements of $P(U_\alpha \cap V_\beta)$, namely $\eta^\alpha | U_\alpha \cap V_\beta$ and ${}'\eta^\alpha | U_\alpha \cap V_\beta$, such that their image under $\Phi(U_\alpha \cap V_\beta)$ is the same. In virtue of theorem 2.5 there exists an open covering $\{W_\gamma\}_\gamma$ of $U_\alpha \cap V_\beta$ such that $\eta^\alpha | W_\gamma = {}'\eta^\beta | W_\gamma$ for all γ, so particularly $\eta^\alpha_x = {}'\eta^\beta_x$ for all $x \in U_\alpha \cap V_\beta$.

QED.

The generalisation of this process, carried out in the case of an arbitrary n, is as follows: Starting with the functor G : Var → Ens (cf.1.11.) we first define a new functor P : Var → Ens by "including enough roots", then we take the associated sheaf P^a of P and show that P^a is representable.

§3. The functor $\mathcal{P}$; the variety M; the reduction theorem.

From now on we restrict ourselves to the cases $n \geq 3$.
We have $\mathcal{G}$: Var $\to$ Ens such that

$\mathcal{G}(S) = \{cl(\mathcal{E};A_1,\dots,A_n) | (A_1,\dots,A_n) \text{ irreducible}\}$ (1.11)

In view of the definition of irreducibility and II, §1 we have the following equivalences: $(A_1,\dots,A_n)$ irreducible $\Leftrightarrow$ for all closed points $x \in S$, $(A_1(x),\dots,A_n(x))$ is irreducible $\Leftrightarrow$ for all closed points $x \in S$ there exists i,j and k with $1 \leq i < j < k \leq n$ such that $(A_i(x),A_j(x),A_k(x))$ is irreducible $\Leftrightarrow$ for all closed points $x \in S$ there exist $1 \leq i < j < k \leq n$ such that at least one of the following pairs is irreducible: $(A_i(x),A_j(x))$, $(A_i(x),A_k(x))$, $(A_j(x),A_k(x))$ and $(A_i(x),A_i(x)+A_j(x)+A_k(x))$.

Now, if we include "roots" which measure the irreducibility of this pairs we may hope to have "enough" roots.

definition 3.1. We look at "families of irreducible n-tuples of endomorphisms equiped with enough roots", in short rigid families. A rigid family on a variety S is an object of the following form:

$(\mathcal{E};A_1,\dots,A_n;r_{1,2},\dots,r_{ij},\dots,r_{n-1,n};r_{1,2,3},\dots,r_{ijk},\dots,r_{n-2,n-1,n})$

where

(1) $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_S$-modules of rank 2.

(2) $A_i \in \text{End}\,\mathcal{E}$ for all $1 \leq i \leq n$ and $(A_1,\dots,A_n)$ is irreducible.

(3) $r_{ij} \in \Gamma(S,\mathcal{O}_S)$ for all $1\leq i<j\leq n$ such that $r_{ij}^2 = 4\text{Tr}(A_iA_j)^2-4\text{Tr}A_i^2A_j^2$, and $r_{ijk} \in \Gamma(S,\mathcal{O}_S)$ for all $1\leq i<j<k\leq n$ such that

$$r_{ijk}^2 = 4\text{Tr}[A_i(A_i+A_j+A_k)]^2-4\text{Tr}A_i^2(A_i+A_j+A_k)^2.$$

remark: From II.1.3 it follows: if $x \in S$ is a closed point then

$r_{ij}(x) \neq 0 \Leftrightarrow (A_i(x),A_j(x))$ is irreducible.

$r_{ijk}(x) \neq 0 \Leftrightarrow (A_i(x),A_i(x)+A_j(x)+A_k(x))$ is irreducible.

<u>definition 3.2</u>. Two rigid families $(E;A_1,\dots,A_n;\dots r_{ij}\dots;\dots r_{ijk}\dots)$ and $(E';A_1',\dots,A_n';\dots r_{ij}'\dots;\dots r_{ijk}'\dots)$ on a variety S are said to be equivalent iff (a) There exists an isomorphism $T : E \overset{\sim}{\to} E'$ such that $T^{-1}A_i'T = A_i$ for all $1\leq i\leq n$.

(b) $r_{ij}' = r_{ij}$ $(1\leq i<j\leq n)$ and $r_{ijk}' = r_{ijk}$ $(1\leq i<j<k\leq n)$

(Notice that (a) already implies $(r_{ij}')^2 = r_{ij}^2$ and $(r_{ijk}')^2 = r_{ijk}^2$).

$cl(E;A_1,\dots,A_n;\dots r_{ij}\dots;\dots r_{ijk}\dots)$ denotes the equivalence class.

If $f : T \to S$ is a morphism of varieties and $(E;A_1,\dots,A_n;\dots r_{ij}\dots;\dots r_{ijk}\dots)$ is a rigid family on S, the pull-back $(f^*E;f^*A_1,\dots,f^*A_n;\dots f^*r_{ij}\dots;\dots f^*r_{ijk}\dots)$ is obviously a rigid family on T. This pull-back respects the given equivalence relation. So we get:

<u>definition 3.3</u>. P : Var $\to$ Ens is the contravariant functor such that, for each variety S, $P(S)$ is the set of equivalence classes of rigid families on S.

<u>Our aim is to prove the representability of the associated sheaf P^a : Var $\to$ Ens</u>. The variety M, which will appear to be a fine moduli space for the functor P^a is defined in 3.4 and 3.5.

<u>definition 3.4</u>. Let M be the polynomial ring in $2n+2\binom{n}{2}+2\binom{n}{3} = \frac{1}{3}n(n^2+5)$ indeterminates X_i $(1\leq i\leq n)$, Y_i $(1\leq i\leq n)$, Z_{ij} $(1\leq i<j\leq n)$, W_{ijk} $(1\leq i<j<k\leq n)$, R_{ij} $(1\leq i<j\leq n)$ and R_{ijk} $(1\leq i<j<k\leq n)$. Hence $M = \Sigma[\dots R_{ij}\dots,\dots R_{ijk}\dots]$. In Σ we have the ideal J (IV.1.8). Define the ideal J_0 in M to be $J_0 = JM+R_0$ where R_0 is the ideal of M generated by

$$R_{ij}^2-(Z_{ij}^2-Y_iY_j) \qquad (1\le i<j\le n)$$
$$R_{ijk}^2-(Z_{ij}^2-Y_iY_j+Z_{ik}^2-Y_iY_k+2Z_{ij}Z_{ik}-2Y_iZ_{jk}) \qquad (1\le i<j<k\le n)$$

For each $1 \le i < j < k \le n$, $\mathcal{B}_{ijk}$ is the ideal in $\mathcal{M}$ generated by R_{ij}, R_{ik}, R_{jk} and R_{ijk}. Finally $\mathcal{B}_0 = \sum_{1\le i<j<k\le n} \mathcal{B}_{ijk}$.

<u>definition 3.5</u>. M is the subvariety of Spec $\mathcal{M} = \mathbb{A}^{\frac{1}{3}n(n^2+5)}$ defined by $M = V(\mathcal{J}_0)-V(\mathcal{B}_0)$. β denotes the inclusion $M \hookrightarrow$ Spec $\mathcal{M}$. For each $1 \le i < j < k \le n$, $M_{ijk} = V(\mathcal{J}_0)-V(\mathcal{B}_{ijk}) \subset M$. The M_{ijk} are open subvarieties of M and $M = \cup M_{ijk}$.

Suppose S is a variety and $\mathrm{cl}(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in\mathcal{P}(S)$. Then we have a morphism $S \to$ Spec $\mathcal{M}$, determined by the k-algebra homomorphism $\mathcal{M} \to \Gamma(S,\mathcal{O}_S)$ such that:

$$\text{3.6}\begin{cases} X_i & \mapsto \mathrm{Tr}A_i \\ Y_i & \mapsto 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 \\ Z_{ij} & \mapsto 2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j \\ W_{ijk} & \mapsto \mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j \\ R_{ij} & \mapsto r_{ij} \\ R_{ijk} & \mapsto r_{ijk} \end{cases}$$

In this way we obtain a morphism of functors $\Psi : \mathcal{P} \to h_{\mathrm{Spec}\,\mathcal{M}}$.

<u>proposition 3.7</u>. If S is a variety and $\xi \in \mathcal{P}(S)$ then the image of $\Psi(S)\xi$ is contained in M.

<u>proof</u>: Let $\xi = \mathrm{cl}(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)$ and let $x \in S$ be a closed point. Then $[\Psi(S)\xi](x)$ is the closed point of Spec $\mathcal{M}$ corresponding to $y = (i(A_1(x),\ldots,A_n(x));\ldots r_{ij}(x)\ldots;\ldots r_{ijk}(x)\ldots)$ where $i(A_1(x),\ldots,A_n(x))$ are the invariants of $(A_1(x),\ldots,A_n(x))$ (cf. I.1.9). $y \in V(\mathcal{J}\mathcal{M})$ because $i(A_1(x),\ldots,A_n(x)) \in V(\mathcal{J})$.

$y \in V(R_0)$ because in general:

$$4[\mathrm{Tr}(AB)^2-\mathrm{Tr}A^2B^2]=[2\mathrm{Tr}AB-\mathrm{Tr}A\mathrm{Tr}B]^2-[2\mathrm{Tr}A^2-(\mathrm{Tr}A)^2][2\mathrm{Tr}B^2-(\mathrm{Tr}B)^2].$$

Therefore $y \in V(J_0)$. From the remarks in the beginning of this section it follows that there exist i,j and k such that $y \notin V(B_{ijk})$. Particularly $y \notin V(B_0) = \cap V(B_{ijk})$ and $y \in M = V(J_0)-V(B_0)$.

<u>definition 3.8</u>. In virtue of the above proposition we have a morphism of functors $\Phi : P \to h_M$ such that

$$\begin{tikzcd} P \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & h_M \arrow[dl, "\beta_*"] \\ & h_{\mathrm{Spec}\ M} & \end{tikzcd}$$

commutes.

The functor P has some interesting subfunctors.

<u>definition 3.9</u>. For all $1 \leq i < j < k \leq n$, P_{ijk} : Var $\to$ Ens is the subfunctor of P such that

$P_{ijk}(S) = \{\mathrm{cl}(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots) \in P(S) | (A_i,A_j,A_k)$ irreducible$\}$.

Notice that P_{ijk} is indeed a subfunctor of P, i.e., if $f : X \to Y$ is a morphism of varieties and $\eta \in P_{ijk}(Y)$ then the pull-back $P(f)\eta$ belongs to $P_{ijk}(X)$.

<u>proposition 3.10</u>. Let S be a variety and $\xi \in P(S)$. Then $\xi \in P_{ijk}(S)$ iff $[\Phi(S)\xi](S) \subset M_{ijk}$.

<u>proof</u>: Let $\xi = \mathrm{cl}(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots) \in P(S)$. Then we have:

$\xi \in P_{ijk}(S) \Leftrightarrow (A_i,A_j,A_k)$ is irreducible

$\Leftrightarrow$ for all closed points $x \in S$, $(A_i(x),A_j(x),A_k(x))$ is irreducible

$\Leftrightarrow$ for all closed points $x \in S$ at least one of the elements of the set $\{r_{ij}(x),r_{ij}(x),r_{jk}(x),r_{ijk}(x)\}$ is different from zero

$\Leftrightarrow [\Phi(S)\xi](S) \subset M_{ijk} = V(J_0)-V(R_{ij},R_{ik},R_{jk},R_{ijk})$.

<u>corollary 3.11</u>. There is a morphism of functors $\Phi_{ijk} : P_{ijk} \to h_{M_{ijk}}$ defined by the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\Phi"] & h_M \\ P_{ijk} \arrow[u, hook] \arrow[r, "\Phi_{ijk}"'] & h_{M_{ijk}} \arrow[u, hook] \end{tikzcd}$$

We want to show that $\Phi^a : P^a \to h_M$ is an isomorphism. From 3.11 we get a commutative diagram

$$\begin{tikzcd} P^a \arrow[r, "\Phi^a"] & h_M \\ P^a_{ijk} \arrow[u, hook] \arrow[r, "\Phi^a_{ijk}"'] & h_{M_{ijk}} \arrow[u, hook] \end{tikzcd}$$

The significance of the subfunctors P_{ijk} will be clear when looking at the following general theorems (3.12 and 3.14).

<u>theorem 3.12</u>. Let $F : \mathrm{Var} \to \mathrm{Ens}$ be a representable sheaf; $\Phi : F \xrightarrow{\sim} h_X$. Suppose Y is a subvariety of X, $\alpha : Y \hookrightarrow X$ the inclusion, and G is a subfunctor of F such that the following condition holds:

(*) If $\xi \in F(S)$ then $\xi \in G(S) \leftrightarrow [\Phi(S)\xi](S) \subset Y$.

Then G is also representable (by Y).

<u>proof</u>: Let $\lambda : G \hookrightarrow F$. Because of (*) we have a morphism of functors $\Psi : G \to h_Y$ such that the diagram below commutes.

$$\begin{tikzcd} F \arrow[r, "\Phi"] & h_X \\ G \arrow[u, hook, "\lambda"] \arrow[r, "\Psi"] & h_Y \arrow[u, hook, "\alpha_*"'] \end{tikzcd}$$

$\Phi(X) : F(X) \to \mathrm{Hom}(X,X)$ is bijective. Hence there exists $\xi_0 \in F(X)$ with

$\Phi(X)\xi_0 = id_X$. $\alpha : Y \hookrightarrow X$ gives $F(\alpha) : F(X) \to F(Y)$. Let $\eta_0 = F(\alpha)\xi_0 \in F(Y)$. I claim: $\eta_0 \in G(Y)$ and $\Psi(Y)\eta_0 = id_Y$ (from which it follows immediately that Ψ is an epimorphism).

Well, in view of the commutativity of the adjacent diagram we have:

$\Phi(Y)\eta_0 = \Phi(Y)F(\alpha)\xi_0 = \alpha^*\Phi(X)\xi_0 = \alpha^*(id_X) = \alpha$.

Hence $[\Phi(Y)\eta_0](Y) = \alpha(Y) = Y$ and thus $\eta_0 \in G(Y)$.

$$\begin{tikzcd} F(X) \arrow[r, "\Phi(X)"] \arrow[d, "F(\alpha)"'] & \mathrm{Hom}(X,X) \arrow[d, "\alpha^*"] \\ F(Y) \arrow[r, "\Phi(Y)"'] & \mathrm{Hom}(Y,X) \end{tikzcd}$$

$\alpha_* : h_Y \to h_X$ is a monomorphism and $\alpha_*\Psi(Y)\eta_0 = \Phi(Y)\eta_0 = \alpha = \alpha_*(id_Y)$, so $\Psi(Y)\eta_0 = id_Y$.

Ψ is also a monomorphism, for suppose $\xi,\xi' \in G(S)$ such that $\Psi(S)\xi = \Psi(S)\xi'$. Then $\alpha_*\Psi(S)\xi = \alpha_*\Psi(S)\xi'$. Hence $\Phi(S)\lambda(S)\xi = \Phi(S)\lambda(S)\xi'$. Since $\Phi(S)$ is bijective and $\lambda(S)$ is injective, it follows that $\xi = \xi'$.

<u>corollary 3.13</u>. If $\Phi^a : P^a \to h_M$ is an isomorphism, then $\Phi^a_{ijk} : P^a_{ijk} \to h_{M_{ijk}}$ is an isomorphism for all $1 \leq i < j < k \leq n$.

<u>proof</u>: We only have to check the condition (*). Let $\omega : P \to P^a$ be the canonical morphism. Let $\xi \in P^a(S)$ be represented by $\{(U_\alpha,\xi^\alpha)\}_{\alpha\in I}$, i.e., $\{U_\alpha\}_{\alpha\in I}$ is an open covering of S and $\xi^\alpha \in P(U_\alpha)$ for all $\alpha \in I$.

$$\begin{aligned}[\Phi^a(S)\xi](S) \subset M_{ijk} &\Leftrightarrow \forall\alpha\in I : [\Phi^a(S)\xi](U_\alpha) \subset M_{ijk} \\ &\Leftrightarrow \forall\alpha\in I : [\Phi^a(U_\alpha)(\xi|U_\alpha)](U_\alpha) \subset M_{ijk} \\ &\Leftrightarrow \forall\alpha\in I : [\Phi^a(U_\alpha)\omega(U_\alpha)\xi^\alpha](U_\alpha) \subset M_{ijk} \\ &\Leftrightarrow \forall\alpha\in I : [\Phi(U_\alpha)\xi^\alpha](U_\alpha) \subset M_{ijk} \\ &\Leftrightarrow \forall\alpha\in I : \xi^\alpha \in P_{ijk}(U_\alpha) \text{ (in view of prop. 3.10)} \\ &\Leftrightarrow \xi \in P^a_{ijk}(S).\end{aligned}$$

The converse of corollary 3.13 is also true as is shown by the following theorem and its corollary.

<u>theorem 3.14</u>. <u>The reduction theorem</u>.

Let F : Var $\to$ Ens be a sheaf, X a variety and $\Phi : F \to h_X$ a morphism of functors. Let $\{X_i\}_{i\in I}$ be an open covering of X and $\alpha_i : X_i \hookrightarrow X$ the inclusion map. For each $i \in I$ let $\mathcal{F}_i$: Var $\to$ Ens be a subsheaf of $\mathcal{F}$ – say $\lambda_i : \mathcal{F}_i \to \mathcal{F}$ – and $\Phi_i : \mathcal{F}_i \to h_{X_i}$ a morphism such that $\Phi \circ \lambda_i = \alpha_{i*} \circ \Phi_i$. Suppose that the following condition holds true for all $i \in I$: If $\xi \in \mathcal{F}(S)$ then $\xi \in \mathcal{F}_i(S)$ iff $[\Phi(S)\xi](S) \subset X_i$ (*).

Then we have: Φ is an isomorphism $\Leftrightarrow$ Φ_i is an isomorphism for all $i \in I$.

<u>proof</u>:

($\Rightarrow$) Evident in view of th.3.12.

($\Leftarrow$) We first prove that Φ is a monomorphism.

Well, let S be a variety and $\xi, \eta \in \mathcal{F}(S)$ such that $\Phi(S)\xi = \Phi(S)\eta = f$.

Let $S_i = f^{-1}(X_i)$ and $\beta_i : S_i \hookrightarrow S$. Since $\{X_i\}_{i\in I}$ is an open covering of X, $\{S_i\}_{i\in I}$ is an open covering of S.

$\xi|S_i = \mathcal{F}(\beta_i)\xi \in \mathcal{F}(S_i)$.

$\Phi(S_i)(\xi|S_i) = \beta_i^*\Phi(S)\xi = \beta_i^*(f) = f \circ \beta_i \in \mathrm{Hom}(S_i,X)$.

Since $(f \circ \beta_i)(S_i) \subset X_i$ it follows from (*) that $\xi|S_i \in \mathcal{F}_i(S_i)$.

Analogously we have $\eta|S_i = \mathcal{F}(\beta_i)\eta \in \mathcal{F}_i(S_i)$.

Using the commutativity of the adjacent diagram we have:

$$\begin{tikzcd}
\mathcal{F}(S) \arrow[r, "\Phi(S)"] \arrow[d, "\mathcal{F}(\beta_i)"'] & \mathrm{Hom}(S,X) \arrow[d, "\beta_i^*"] \\
\mathcal{F}(S_i) \arrow[r, "\Phi(S_i)"] & \mathrm{Hom}(S_i,X) \\
\mathcal{F}_i(S_i) \arrow[u, hook] \arrow[r, "\Phi_i(S_i)"] & \mathrm{Hom}(S_i,X_i) \arrow[u, "\alpha_{i*}"']
\end{tikzcd}$$

$\beta_i^*\Phi(S)\xi = \Phi(S_i)(\xi|S_i) = \alpha_{i*}\Phi_i(S_i)(\xi|S_i)$.

And also $\beta_i^*\Phi(S)\eta = \alpha_{i*}\Phi_i(S_i)(\eta|S_i)$.

Now $\Phi(S)\xi = \Phi(S)\eta$, α_{i*} is a monomorphism and Φ_i is an isomorphism.

Hence $\xi|S_i = \eta|S_i \in \mathcal{F}(S_i)$ for all $i \in I$, from which it follows that $\xi = \eta$, $\mathcal{F}$ being a sheaf.

In order that $\Phi : \mathcal{F} \to h_X$ is an epimorphism we have to show the existence of an element ξ in $\mathcal{F}(X)$ such that $\Phi(X)\xi = id_X$. Since $\Phi_i : \mathcal{F}_i \to h_{X_i}$ is an isomorphism there exists $\eta_i \in \mathcal{F}_i(X_i)$ such that

$\Phi_i(X_i)\eta_i = id_{X_i}$. Put $\xi_i = \lambda_i(X_i)\eta_i \in F(X_i)$.

For all $i,j \in I$, $\xi_i|X_i \cap X_j \in F(X_i \cap X_j)$ and:

$$\begin{aligned}\Phi(X_i \cap X_j)(\xi_i|X_i \cap X_j) &= \Phi(X_i \cap X_j)\lambda_i(X_i \cap X_j)[\eta_i|X_i \cap X_j] = \\ &= \alpha_{i*}\Phi_i(X_i \cap X_j)[\eta_i|X_i \cap X_j] = \\ &= \alpha_{i*}[id_{X_i}|X_i \cap X_j] = \\ &= \text{the composition of } X_i \cap X_j \hookrightarrow X_i \text{ and } X_i \hookrightarrow X.\end{aligned}$$

Similarly: $\Phi(X_i \cap X_j)(\xi_j|X_i \cap X_j)$ = the composition of $X_i \cap X_j \hookrightarrow X_j \hookrightarrow X$.

Therefore $\Phi(X_i \cap X_j)(\xi_i|X_i \cap X_j) = \Phi(X_i \cap X_j)(\xi_j|X_i \cap X_j)$ for all $i,j \in I$.

Hence $\xi_i|X_i \cap X_j = \xi_j|X_i \cap X_j$ because Φ is a monomorphism.

Now F is a sheaf, $\{X_i\}_{i\in I}$ is an open covering of X and $\xi_i \in F(X_i)$ for all $i \in I$ such that $\forall i,j \in I : \xi_i|X_i \cap X_j = \xi_j|X_i \cap X_j$. Therefore there exists $\xi \in F(X)$ such that $\xi|X_i = \xi_i = \lambda_i(X_i)\eta_i$ for all $i \in I$.

Using the commutativity of the adjacent diagram we get, for all $i \in I$:

$$\begin{tikzcd} F(X) \arrow[r, "\Phi(X)"] \arrow[d] & \mathrm{Hom}(X,X) \arrow[d, "\alpha_i^*"] \\ F(X_i) \arrow[r, "\Phi(X_i)"] & \mathrm{Hom}(X_i,X) \\ F_i(X_i) \arrow[u, hook, "\lambda_i(X_i)"] \arrow[r, "\Phi_i(X_i)"] & \mathrm{Hom}(X_i,X_i) \arrow[u, hook, "\alpha_{i*}"'] \end{tikzcd}$$

$$\begin{aligned}[\Phi(X)\xi]|X_i &= \alpha_i^*\Phi(X)\xi \\ &= \Phi(X_i)(\xi|X_i) \\ &= \Phi(X_i)(\xi_i) \\ &= \Phi(X_i)\lambda_i(X_i)\eta_i \\ &= \alpha_{i*}\Phi_i(X_i)\eta_i \\ &= \alpha_{i*}(id_{X_i}) \\ &= id_X|X_i\end{aligned}$$

Since h_X is a sheaf and $X = \bigcup_{i\in I} X_i$, this implies $\Phi(X)\xi = id_X$.

<u>corollary 3.15</u>. $\Phi^a : P^a \to h_M$ is an isomorphism iff $\Phi^a_{ijk} : P^a_{ijk} \to h_{M_{ijk}}$ is an isomorphism for all $1 \leq i < j < k \leq n$.

Now suppose that p, q and r are fixed integers and $1 \leq p<q<r \leq n$.

We carry out another reduction on $\Phi^a_{pqr} : P^a_{pqr} \to h_{M_{pqr}}$. Define four subfunctors of P_{pqr} by:

$\mathcal{Q}_1(S)=\{cl(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in P_{pqr}(S)\,|\,(A_p,A_q) \text{ irreducible}\}$

$\mathcal{Q}_2(S)=\{cl(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in P_{pqr}(S)\,|\,(A_p,A_r) \text{ irreducible}\}$

$\mathcal{Q}_3(S)=\{cl(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in P_{pqr}(S)\,|\,(A_q,A_r) \text{ irreducible}\}$

$\mathcal{Q}_4(S)=\{cl(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in P_{pqr}(S)\,|\,(A_p,A_p+A_q+A_r) \text{ irr.}\}$

Let $N_i \subset M_{pqr}$ be the open subvarieties defined by $N_1 = V(J_0)-V(R_{pq})$, $N_2 = V(J_0)-V(R_{pr})$, $N_3 = V(J_0)-V(R_{qr})$ and $N_4 = V(J_0)-V(R_{pqr})$.
Then $M_{pqr} = N_1 \cup N_2 \cup N_3 \cup N_4$. For all $1 \leq i \leq 4$ and $\xi \in P_{pqr}(S)$ we have:

$$\xi \in \mathcal{Q}_i(S) \Leftrightarrow [\Phi_{pqr}(S)\xi](S) \subset N_i .$$

Hence there exist morphisms $\Phi_i : \mathcal{Q}_i \to h_{N_i}$ such that the diagram

$$\begin{tikzcd} P_{pqr} \arrow[r, "\Phi_{pqr}"] & h_{M_{pqr}} \\ \mathcal{Q}_i \arrow[u, hook] \arrow[r, "\Phi_i"] & h_{N_i} \arrow[u, hook] \end{tikzcd}$$

commutes for all $1 \leq i \leq 4$.

Reasoning in the same way as above and using again the reduction theorem one gets:

<u>proposition 3.16</u>. Φ^a_{pqr} is an isomorphism $\Leftrightarrow$ Φ^a_i is an isomorphism $(1 \leq i \leq 4)$.

Now $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$, viewed as subfunctors of P, are of the same type. $\mathcal{Q}_4$ is another type of subfunctor. Forgetting the various subfunctors we defined so far we sum up the whole process of reduction carried out in this section.

<u>definition 3.17</u>. For all $1 \leq i < j \leq n$ let $\mathcal{Q}_{ij}$: Var $\to$ Ens be the

subfunctor of P defined by:

$\mathcal{Q}_{ij}(S)=\{cl(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in P(S)\,|\,(A_i,A_j)$ irreducible$\}$.

For all $1 \leq i < j < k \leq n$ $\mathcal{Q}_{ijk}$: Var $\to$ Ens is the subfunctor of P such that $\mathcal{Q}_{ijk}(S)=\{cl(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)\in P(S)\,|\,(A_i,A_i+A_j+A_k)$ irreducible$\}$.

Define $N_{ij} = V(J_0)-V(R_{ij}) \subset M \quad (1 \leq i < j \leq n)$

$N_{ijk} = V(J_0)-V(R_{ijk}) \subset M \quad (1 \leq i < j < k \leq n)$

Let $\Phi_{ij} : \mathcal{Q}_{ij} \to h_{N_{ij}}$ and $\Phi_{ijk} : \mathcal{Q}_{ijk} \to h_{N_{ijk}}$ be the morphisms of functors defined by the commutative diagrams:

$$\begin{tikzcd} P \arrow[r, "\Phi"] & h_M \\ \mathcal{Q}_{ij} \arrow[u, hook] \arrow[r, "\Phi_{ij}"'] & h_{N_{ij}} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\Phi"] & h_M \\ \mathcal{Q}_{ijk} \arrow[u, hook] \arrow[r, "\Phi_{ijk}"'] & h_{N_{ijk}} \arrow[u, hook] \end{tikzcd}$$

Combining corollary 3.15 and proposition 3.16 we have proved:

<u>theorem 3.18</u>. $\Phi^a : P^a \to h_M$ is an isomorphism if and only if $\Phi^a_{ij} : \mathcal{Q}^a_{ij} \to h_{N_{ij}}$ and $\Phi^a_{ijk} : \mathcal{Q}^a_{ijk} \to h_{N_{ijk}}$ are isomorphisms for all $1 \leq i < j \leq n$ and $1 \leq i < j < k \leq n$ respectively.

§4. Φ^a_{ij} and Φ^a_{ijk} are monomorphisms.

<u>lemma 4.1</u>. Let F : Var $\to$ Ens be a contravariant functor, X a variety and $\theta : F \to h_X$ a morphism of functors satisfying, for each variety S, the condition:

$$(*)\left\{\begin{array}{l}\text{If } \eta,\eta' \in F(S) \text{ such that } \theta(S)\eta = \theta(S)\eta'\text{, then there}\\ \text{exists an open covering } \{U_\alpha\}_{\alpha\in I} \text{ of S such that}\\ \eta|U_\alpha = \eta'|U_\alpha \in F(U_\alpha) \text{ for all } \alpha \in I.\end{array}\right.$$

Then $\theta^a : F^a \to h_X$ is a monomorphism.

<u>proof</u>: Let S be a variety and $\xi,\xi' \in F^a(S)$ such that $\theta^a(S)\xi = \theta^a(S)\xi'$. We may assume that ξ and ξ' are represented by $\{(U_\alpha,\xi^\alpha)\}_{\alpha\in I}$ and

$\{(U_\alpha,{}'\xi^\alpha)\}_{\alpha\in I}$ respectively. We have to show $\xi = \xi'$, i.e., $\forall\alpha,\beta \in I\ \forall x \in U_\alpha \cap U_\beta\ [\xi^\alpha_x = {}'\xi^\beta_x]$.

Well, take $\alpha,\beta \in I$. Let $\omega : F \to F^a$ be the canonical morphism. The following diagram is commutative.

$$\begin{tikzcd}
F^a(S) \arrow[r, "\theta^a(S)"] \arrow[d] & \mathrm{Hom}(S,X) \arrow[dd] \\
F^a(U_\alpha) \arrow[dr, "\theta^a(U_\alpha)"] & \\
F(U_\alpha) \arrow[u, "\omega(U_\alpha)"] \arrow[r, "\theta(U_\alpha)"] & \mathrm{Hom}(U_\alpha,X)
\end{tikzcd}$$

Hence $[\theta^a(S)\xi]|U_\alpha = \theta^a(U_\alpha)[\xi|U_\alpha] = \theta^a(U_\alpha)\omega(U_\alpha)\xi^\alpha = \theta(U_\alpha)\xi^\alpha$.

Analogously: $[\theta^a(S)\xi']|U_\alpha = \theta(U_\alpha){}'\xi^\alpha$.

Since $\theta^a(S)\xi = \theta^a(S)\xi'$ it follows that $\theta(U_\alpha)\xi^\alpha = \theta(U_\alpha){}'\xi^\alpha$. Using condition (*) there exists an open covering $\{W_\gamma\}_\gamma$ of U_α such that $\xi^\alpha|W_\gamma = {}'\xi^\alpha|W_\gamma$ for all γ, so certainly $\xi^\alpha_x = {}'\xi^\alpha_x$ for all $x \in U_\alpha$. Now ${}'\xi^\alpha_x = {}'\xi^\beta_x$ for all $x \in U_\alpha \cap U_\beta$. Therefore $\xi^\alpha_x = {}'\xi^\beta_x$ for all $x \in U_\alpha \cap U_\beta$.

Looking back at §2 we have the following situation:

$P_2 : \mathrm{Var} \to \mathrm{Ens}$ is defined by $P_2(S) = \{\mathrm{cl}(E;A_1,A_2;r)\,|\,(A_1,A_2)$ irreducible$\}$

$M_2 = V(R^2-Z_{12}^2+Y_1Y_2)-V(R)\subset \mathrm{Spec}\ k[X_1,X_2,Y_1,Y_2,Z_{12},R]$; $\Phi_2 : P_2 \to h_{M_2}$.

<u>theorem 4.2</u>. $\Phi^a_{ij} : Q^a_{ij} \to h_{N_{ij}}$ is a monomorphism for all $1 \leq i < j \leq n$.

<u>proof</u>: Let $u : Q_{ij} \to P_2$ be the morphism of functors defined by

$u(S)[\mathrm{cl}(E;A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)] = \mathrm{cl}(E;A_i,A_j;r_{ij})$.

Let $f' : k[X_1,X_2,Y_1,Y_2,Z_{12},R] \to M = \Sigma[\ldots R_{ij}\ldots,\ldots R_{ijk}\ldots]$ be the k-algebra homomorphism such that $f'(X_1) = X_i$, $f'(X_2) = X_j$, $f'(Y_1) = Y_i$, $f'(Y_2) = Y_j$, $f'(Z_{12}) = Z_{ij}$ and $f'(R) = R_{ij}$. f' induces a morphism $f : \mathrm{Spec}\ M \to \mathrm{Spec}\ k[X_1,X_2,Y_1,Y_2,Z_{12},R]$ and from the definition of f', M_2 and N_{ij} it follows that f induces a morphism $g : N_{ij} \to M_2$.

The diagram below is commutative.

$$\begin{tikzcd}
& \mathcal{P} \arrow[drrr, "\Psi"] & & & \\
\mathcal{Q}_{ij} \arrow[ur, hook, "\lambda"] \arrow[rr, "\Phi_{ij}"'] \arrow[d, "u"'] & & h_{N_{ij}} \arrow[rr, hook] \arrow[d, "g_*"] & & h_{\mathrm{Spec}\ \mathcal{M}} \arrow[d, "f_*"] \\
\mathcal{P}_2 \arrow[rr, "\Phi_2"'] & & h_{M_2} \arrow[rr, hook, "\mu"'] & & h_{\mathrm{Spec}\ k[X_1,X_2,Y_1,Y_2,Z_{12},R]}
\end{tikzcd}$$

The only non-trivial thing to prove is: $f_*\Psi\lambda = \mu\Phi_2 u$.

Well, let S be a variety and

$\xi = \mathrm{cl}(\mathcal{E};A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots) \in \mathcal{Q}_{ij}(S)$.

Then $\mu(S)\Phi_2(S)u(S)\xi = \mu(S)\Phi_2(S)[\mathrm{cl}(\mathcal{E};A_i,A_j;r_{ij})]$ is the morphism from S to Spec $k[X_1,X_2,Y_1,Y_2,Z_{12},R]$ determined by the k-algebra homomorphism $k[X_1,X_2,Y_1,Y_2,Z_{12},R] \to \Gamma(S,\mathcal{O}_S)$ such that

$$\begin{array}{llllll}
X_1 \mapsto \mathrm{Tr}A_i & & Y_1 \mapsto 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 & & Z_{12} \mapsto 2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j \\
X_2 \mapsto \mathrm{Tr}A_j & & Y_2 \mapsto 2\mathrm{Tr}A_j^2-(\mathrm{Tr}A_j)^2 & & R \mapsto r_{ij}
\end{array}$$

Taking into account the definition of f above and of $\Psi : \mathcal{P} \to h_{\mathrm{Spec}\ \mathcal{M}}$ this implies $\mu(S)\Phi_2(S)u(S)\xi = f_*(S)\Psi(S)\lambda(S)\xi$.

In order to prove that Φ^a_{ij} is a monomorphism, we want to apply lemma 4.1. So we have to show:

> If $\eta,\eta' \in \mathcal{Q}_{ij}(S)$ and $\Phi_{ij}(S)\eta = \Phi_{ij}(S)\eta'$ then there exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of S such that $\eta|U_\alpha = \eta'|U_\alpha \in \mathcal{Q}_{ij}(U_\alpha)$ for all $\alpha \in I$.

Say $\eta = \mathrm{cl}(\mathcal{E};A_1,\ldots,A_n;\ldots r_{ij}\ldots;\ldots r_{ijk}\ldots)$ and

$\eta' = \mathrm{cl}(\mathcal{E}';A'_1,\ldots,A'_n;\ldots r'_{ij}\ldots;\ldots r'_{ijk}\ldots)$.

$\Phi_{ij}(S)\eta = \Phi_{ij}(S)\eta' \Rightarrow \Psi(S)\lambda(S)\eta = \Phi(S)\lambda(S)\eta'$.

Hence for all closed points $x \in S$ we have:

$$i(A_1(x),\dots,A_n(x)) = i(A'_1(x),\dots,A'_n(x))$$

$$r_{ab}(x) = r'_{ab}(x)) \quad (1 \le a < b \le n)$$

$$r_{abc}(x) = r'_{abc}(x) \ (1 \le a < b < c \le n)$$

In particular this implies: $r_{ab} = r'_{ab}$ for all $1 \le a < b \le n$ and $r_{abc} = r'_{abc}$ for all $1 \le a < b < c \le n$ (1).

Let $\xi = u(S)\eta = cl(E;A_i,A_j;r_{ij})$ and $\xi' = u(S)\eta' = cl(E';A'_i,A'_j;r'_{ij})$. Then $\Phi_2(S)\xi = \Phi_2(S)u(S)\eta = g_*(S)\Phi_{ij}(S)\eta = g_*(S)\Phi_{ij}(S)\eta' = \Phi_2(S)\xi'$. In virtue of theorem 2.5 there exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of S such that $\xi|U_\alpha = \xi'|U_\alpha \in P_2(U_\alpha)$ for all $\alpha \in I$. In particular we have isomorphisms $T_\alpha : E|U_\alpha \to E'|U_\alpha$ such that the diagrams

$$\begin{tikzcd} E|U_\alpha \arrow[r, "T_\alpha"] \arrow[d, "A_i|U_\alpha"'] & E'|U_\alpha \arrow[d, "A'_i|U_\alpha"] \\ E|U_\alpha \arrow[r, "T_\alpha"] & E'|U_\alpha \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E|U_\alpha \arrow[r, "T_\alpha"] \arrow[d, "A_j|U_\alpha"'] & E'|U_\alpha \arrow[d, "A'_j|U_\alpha"] \\ E|U_\alpha \arrow[r, "T_\alpha"] & E'|U_\alpha \end{tikzcd}$$

commute. Eventually restricting the U_α we may suppose that $E|U_\alpha \simeq \mathcal{O}^2_{U_\alpha}$ for all $\alpha \in I$. Choosing a basis in $E|U_\alpha$ we can assume that $A_p|U_\alpha$ and $T_\alpha^{-1}(A'_p|U_\alpha)T_\alpha$ being endomorphisms of $E|U_\alpha$ are 2×2 matrices with entries in $\Gamma(U_\alpha,\mathcal{O}_S)$ for all $1 \le p \le n$. For each closed point x in U_α we then have:

$$i(A_1(x),\dots,A_n(x))=i(A'_1(x),\dots,A'_n(x)) =$$
$$= i([T_\alpha^{-1}(A'_1|U_\alpha)T_\alpha](x),\dots,[T_\alpha^{-1}(A'_n|U_\alpha)T_\alpha](x)).$$

Since $[T_\alpha^{-1}(A'_i|U_\alpha)T_\alpha](x) = A_i(x)$, $[T_\alpha^{-1}(A'_j|U_\alpha)T_\alpha](x) = A_j(x)$ and since the pair $(A_i(x),A_j(x))$ is irreducible we must have, in consequence of corollary II.2.5, for <u>all</u> $1 \le p \le n$: $[T_\alpha^{-1}(A'_p|U_\alpha)T_\alpha](x) = A_p(x)$.

Hence $T_\alpha^{-1}(A'_p|U_\alpha)T_\alpha = A_p|U_\alpha \in End(E|U_\alpha)$ for all $\alpha \in I$ and $1 \le p \le n$ (2).

Combining (1) and (2) we have $\eta|U_\alpha = \eta'|U_\alpha$ for all $\alpha \in I$. QED.

<u>theorem 4.3</u>. $\Phi^a_{ijk} : \mathcal{Q}^a_{ijk} \to h_{N_{ijk}}$ is a monomorphism for all $1 \le i<j<k \le n$.

<u>proof</u>: Analogous to th.4.2 using the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{Q}_{ijk} \arrow[r, "\Phi_{ijk}"] \arrow[d, "v"'] & h_{N_{ijk}} \arrow[r, hook] \arrow[d] & h_{\mathrm{Spec}\ \mathcal{M}} \arrow[d, "f_*"] \\
\mathcal{P}_2 \arrow[r, "\Phi_2"'] & h_{\mathcal{M}_2} \arrow[r, hook] & h_{\mathrm{Spec}\ k[X_1,X_2,Y_1,Y_2,Z_{12},R]}
\end{tikzcd}$$

where $v(S)cl(E;A_1,\dots,A_n;\dots r_{ij}\dots;\dots r_{ijk}\dots)=cl(E;A_i,A_i+A_j+A_k;r_{ijk})$ and $f : \mathrm{Spec}\ \mathcal{M} \to \mathrm{Spec}\ k[X_1,X_2,Y_1,Y_2,Z_{12},R]$ is determined by the k-algebra homomorphism $f' : k[X_1,X_2,Y_1,Y_2,Z_{12},R] \to \mathcal{M}$ such that

$f'(X_1) = X_i$, $f'(X_2) = X_i+X_j+X_k$, $f'(Y_1) = Y_i$,

$f'(Y_2) = Y_i+Y_j+Y_k+2Z_{ij}+2Z_{ik}+2Z_{jk}$, $f'(Z_{12}) = Y_i+Z_{ij}+Z_{ik}$ and

$f'(R) = R'_{ijk}$.

§5. <u>Φ_{ij} and Φ_{ijk} are epimorphisms</u>.

$\Phi_{ij} : \mathcal{Q}_{ij} \to h_{N_{ij}}$ and $\Phi_{ijk} : \mathcal{Q}_{ijk} \to h_{N_{ijk}}$ are epimorphisms if and only if there exist elements $\xi_{ij} \in \mathcal{Q}_{ij}(N_{ij})$ and $\xi_{ijk} \in \mathcal{Q}_{ijk}(N_{ijk})$ such that $\Phi_{ij}(N_{ij})\xi_{ij} = id_{N_{ij}}$ and $\Phi_{ijk}(N_{ijk})\xi_{ijk} = id_{N_{ijk}}$, respectively.

We construct ξ_{ij} and ξ_{ijk} with the aid of a "standard model". Let us first explain what is meant by "standard model". We go all the way back to §1. There we have defined contravariant functors F and G, where $F(S) = \{cl(E;A_1,\dots,A_n) | E$ locally free of rank 2, $A_i \in \mathrm{End}\ E\}$ and $G(S) = \{cl(E;A_1,\dots,A_n) \in F(S) | (A_1,\dots,A_n)$ irreducible$\}$. Furthermore, we had a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\Phi"] & h_Y \\
G \arrow[u, hook] \arrow[r, "\Phi'"'] & h_{Y_s} \arrow[u, hook]
\end{tikzcd}.$$

Identifying Y with Spec $^{\Sigma}/J$, $\Phi(S)\mathrm{cl}(E;A_1,\ldots,A_n) \in \mathrm{Hom}(S,\mathrm{Spec}\ ^{\Sigma}/J)$ is determined by the k-algebra homomorphism $^{\Sigma}/J \to \Gamma(S,\mathcal{O}_S)$ such that

$$X_i+J \mapsto \mathrm{Tr}A_i$$
$$Y_i+J \mapsto 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2$$
$$Z_{ij}+J \mapsto 2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j$$
$$W_{ijk}+J \mapsto \mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j$$

We shall define an affine variety N_0 and a morphism $p : N_0 \to Y$. After that we construct the "standard model" (N_0,ξ_0), i.e., $\xi_0 \in F(N_0)$ such that $\Phi(N_0)\xi_0 = p$. Moreover ξ_0 will appear to be in $G(N_0)$.

Before defining N_0 explicitly we need some algebra.

<u>lemma 5.1</u>. Let A be a ring, B an ideal of A, $\pi : A \to {}^{A}/_{B}$ the canonical projection and $r \in A$. Let B^e denote the extension of B in A[T], i.e., $B^e = BA[T] = \{a_0+a_1T+\ldots+a_mT^m \mid m \in \mathbb{N}_0,\ a_i \in B \text{ for all } 0 \le i \le m\}$. Then there is a canonical isomorphism ${}^{A[T]}/_{B^e+(T^2-r)} \xrightarrow{\sim} {}^{A/B[T]}/_{(T^2-\pi(r))}$.

<u>proof</u>:

Define the homomorphism $\bar{\pi} : A[T] \to {}^{A}/_{B}[T]$ by $\bar{\pi}(\sum_{i=0}^{m} a_iT^i) = \sum_{i=0}^{m} \pi(a_i)T^i$. Then $B^e = \ker \bar{\pi}$. Let θ denote the composition of $\bar{\pi}$ and the canonical projection ${}^{A}/_{B}[T] \to {}^{A/B[T]}/_{(T^2-\pi(r))}$, then θ is surjective and the following equivalences hold true:

$$\sum_{i=0}^{m} a_iT^i \in \ker\theta \Leftrightarrow \sum_{i=0}^{m} \pi(a_i)T^i \in (T^2-\pi(r))$$
$$\Leftrightarrow \text{there exist } b_j \in A \text{ such that } \sum_{i=0}^{m} \pi(a_i)T^i = (T^2-\pi(r))\sum_{j=0}^{n} \pi(b_j)T^j$$
$$\Leftrightarrow \bar{\pi}[\sum_{i=0}^{m} a_iT^i-(T^2-r)\sum_{j=0}^{n} b_jT^j] = 0 \text{ for some } b_j \in A$$
$$\Leftrightarrow \sum_{i=0}^{m} a_iT^i \in B^e+(T^2-r).$$

Hence θ induces an isomorphism $A[T]/_{\mathcal{B}^e+(T^2-r)} \overset{\sim}{\to} {}^{A}/_{\mathcal{B}}[T]/_{(T^2-\pi(r))}$.

lemma 5.2. Let B be an integral domain and $a \in B-\{0\}$ such that $\forall b_1,b_2 \in B[b_1^2=b_2^2a \Leftrightarrow b_1=b_2=0]$. Then (T^2-a) is a prime ideal in $B[T]$.

proof: Let $x,y \in B[T]/_{(T^2-a)}$, $x \neq 0$ and $xy = 0$. Say $x = b_1+b_2T+(T^2-a)$ and $y = c_1+c_2T+(T^2-a)$. Now $xy=0 \Leftrightarrow b_1c_1+b_2c_2a=b_1c_2+b_2c_1=0$ (1).

(1) implies $0 = b_1b_2c_1+b_2^2c_2a = b_1b_2c_1+b_1^2c_2$. Hence $c_2(b_2^2a-b_1^2) = 0$.

Since B is an integral domain and $x \neq 0$ we have $c_2 = 0$. In view of (1) it follows that $b_1c_1 = b_2c_1 = 0$ and thus $c_1 = 0$. Hence $y = 0$.

corollary 5.3. Let A, $\mathcal{B}$, π and $\mathcal{B}^e$ be as in 5.1. Suppose that $\mathcal{B}$ is a prime ideal and $r \in A-\mathcal{B}$. Finally assume $\forall p,q \in A[p^2-q^2r \in \mathcal{B} \Rightarrow p,q \in \mathcal{B}]$. Then $\mathcal{B}^e+(T^2-r)$ is a prime ideal in $A[T]$.

proof: Evident in view of the above lemmas.

definition 5.4. $\Sigma \subset \Sigma[R]$. In Σ we have the prime ideal J (cf. IV.1.8.) Let J_1 be the ideal in $\Sigma[R]$ defined by $J_1 = J\Sigma[R]+(R^2-Z_{12}^2+Y_1Y_2)$.

proposition 5.5. J_1 is a prime ideal in $\Sigma[R]$.

proof: Apply cor.5.3. with $A = \Sigma$, $\mathcal{B} = J$ and $r = Z_{12}^2-Y_1Y_2$. It remains to show that $\forall p,q \in \Sigma[\alpha(p^2-q^2r) = 0 \Rightarrow \alpha(p)=\alpha(q)=0]$.

Let $p,q \in \Sigma$ such that $\alpha(p)^2=\alpha(q)^2\alpha(r)$ in $R = k[\underline{X}_1,\ldots,\underline{X}_n]$ and suppose $\alpha(q) \neq 0$. Since $\alpha(r) = 4[\mathrm{Tr}(\underline{X}_1\ \underline{X}_2)^2-\mathrm{Tr}\underline{X}_1^2\underline{X}_2^2] \in k[\underline{X}_1,\underline{X}_2]$ there exist also $P,Q \in k[\underline{X}_1,\underline{X}_2]$ such that $Q \neq 0$ and $Q^2\alpha(r) = P^2$. Now we may assume P and Q to be relatively prime which implies $Q = \pm 1$. Hence there exists $\gamma(\underline{X}_1,\underline{X}_2) \in k[\underline{X}_1,\underline{X}_2]$ such that $\mathrm{Tr}(\underline{X}_1\underline{X}_2)^2-\mathrm{Tr}\underline{X}_1^2\underline{X}_2^2 = \gamma(\underline{X}_1,\underline{X}_2)^2$ Substituting $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ for $\underline{X}_2$ we obtain that $-X_{1;12}X_{1;21}$ is a square in $k[\underline{X}_1]$. Contradiction. So $\alpha(q) = 0$ whence $\alpha(p) = 0$ too.

definition 5.6. Let N be the irreducible affine variety defined by $N = \mathrm{Spec}\ \Sigma[R]/J_1$. Let $N_0 = N_{R+J_1} = \mathrm{Spec}\left(\Sigma[R]/J_1\right)_{R+J_1}$. $p : N_0 \to Y$ is the morphism determined by the composition of the canonical homomorphisms $\Sigma/J \to \Sigma[R]/J_1 \hookrightarrow \left(\Sigma[R]/J_1\right)_{R+J_1}$. Finally we define two open subsets of N_0 by $U_1 = N_0 \cap N_{R-Z_{12}+J_1} = N_{R(R-Z_{12})+J_1}$ and $U_2 = N_0 \cap N_{R+Z_{12}+J_1} = N_{R(R+Z_{12})+J_1}$. Then $N_0 = U_1 \cup U_2$.

<u>theorem 5.7</u>. There exists $\xi_1 = cl(\mathcal{O}^2_{U_1};A_1,\dots,A_n) \in F(U_1)$ satisfying $\Phi(U_1)\xi_1 = p|U_1 \in \mathrm{Hom}(U_1,Y)$.

<u>proof</u>:

$\Gamma(U_1,\mathcal{O}_{U_1}) = \left(\Sigma[R]/J_1\right)_{R(R-Z_{12})+J_1} \cong \Sigma[R]_{R(R-Z_{12})}\Big/ S^{-1}J_1$ where S is the multiplicatively closed subset $\{R^m(R-Z_{12})^m | m \geq 0\}$ of $\Sigma[R]$.

Define elements of $M(2,\Sigma[R]_{R(R-Z_{12})})$ as follows (cf. proof of prop. 2.1). If $1 \leq i \leq n$ then

$$A_i = \tfrac{1}{2}\begin{pmatrix} X_i-2R^{-1}W_{12i} & R^{-1}(Z_{12}-R)^{-1}[Y_2Z_{1i}-Z_{2i}(Z_{12}-R)] \\ R^{-1}[Y_1Z_{2i}-Z_{1i}(Z_{12}-R)] & X_i+2R^{-1}W_{12i} \end{pmatrix}$$

where $Z_{11} = Y_1$, $Z_{22} = Y_2$, $Z_{21} = Z_{12}$ and $W_{121} = W_{122} = 0$, or explicitly:

$$A_1 = \tfrac{1}{2}\begin{pmatrix} X_1 & 1 \\ Y_1 & X_1 \end{pmatrix} \text{ and } A_2 = \tfrac{1}{2}\begin{pmatrix} X_2 & Y_2(Z_{12}-R)^{-1} \\ Z_{12}-R & X_2 \end{pmatrix}.$$

Because of the above isomorphism we can consider these matrices elements of $M(2,\Gamma(U_1,\mathcal{O}_{U_1}))$. Define $\xi_1 = cl(\mathcal{O}^2_{U_1};A_1,\dots,A_n) \in F(U_1)$.

$p|U_1$ is determined by the composition of the canonical homomorphisms $\Sigma/J \to \Sigma[R]/J_1 \hookrightarrow \left(\Sigma[R]/J_1\right)_{R(R-Z_{12})+J_1}$. Hence $\Phi(U_1)\xi_1 = p|U_1$ iff in $\left(\Sigma[R]/J_1\right)_{R(R-Z_{12})+J_1}$ we have

$$\begin{cases} (1)\ \mathrm{Tr}A_i = X_i & (1\leq i\leq n) \\ (2)\ 2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 = Y_i & (1\leq i\leq n) \\ (3)\ 2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j = Z_{ij} & (1\leq i<j\leq n) \\ (4)\ \mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j = W_{ijk} & (1\leq i<j<k\leq n). \end{cases}$$

In checking these identities we write "=" even if we work modulo $S^{-1}J_1$.

For all $1\leq i\leq n$ we put $a_i=X_i-2R^{-1}W_{12i}$, $b_i=R^{-1}(Z_{12}-R)^{-1}[Y_2Z_{1i}-Z_{2i}(Z_{12}-R)]$, $c_i=R^{-1}[Y_1Z_{2i}-Z_{1i}(Z_{12}-R)]$ and $d_i=X_i+2R^{-1}W_{12i}$; hence $A_i = \frac{1}{2}\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

Remark: $(Z_{12}-R)(Z_{12}+R) = Z_{12}^2-R^2 = Y_1Y_2$ (*).

(1) Evidently $\mathrm{Tr}A_i = X_i$ for all i.

(2) $2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 = \frac{1}{4}(a_i-d_i)^2+b_ic_i = R^{-2}(4W_{12i}^2+R^2b_ic_i)$.

$b_1c_1 = Y_1$ and $b_2c_2 = Y_2$, so the identity holds true if $i = 1,2$. If $i \geq 3$ one has, in view of (*):

$$\begin{aligned} R^2b_ic_i &= (Z_{12}-R)^{-1}Y_1Y_2Z_{1i}Z_{2i}-Y_1Z_{2i}^2-Y_2Z_{1i}^2+(Z_{12}-R)Z_{1i}Z_{2i} \\ &= -Y_1Z_{2i}^2-Y_2Z_{1i}^2+2Z_{12}Z_{1i}Z_{2i}. \end{aligned}$$

Hence $2\mathrm{Tr}A_i^2-(\mathrm{Tr}A_i)^2 = R^{-2}(\pi f(1,2,i)+Y_iZ_{12}^2-Y_1Y_2Y_i) = Y_i$.

(3) $2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j = \frac{1}{4}(a_i-d_i)(a_j-d_j)+\frac{1}{2}(b_ic_j+c_ib_j)$.

$$\begin{aligned} (b_ic_j+c_ib_j)R^2 &= (Z_{12}-R)^{-1}[Y_2Z_{1i}-Z_{2i}(Z_{12}-R)][Y_1Z_{2j}-Z_{1j}(Z_{12}-R)]+ \\ &\quad +(Z_{12}-R)^{-1}[Y_2Z_{1j}-Z_{2j}(Z_{12}-R)][Y_1Z_{2i}-Z_{1i}(Z_{12}-R)] = \\ &= Z_{1i}Z_{2j}(Z_{12}+R)-2Y_2Z_{1i}Z_{1j}-2Y_1Z_{2i}Z_{2j}+Z_{1j}Z_{2i}(Z_{12}-R)+ \\ &\quad +Z_{1j}Z_{2i}(Z_{12}+R)+Z_{1i}Z_{2j}(Z_{12}-R) = \\ &= 2(Z_{12}Z_{1i}Z_{2j}+Z_{12}Z_{1j}Z_{2i}-Y_1Z_{2i}Z_{2j}-Y_2Z_{1i}Z_{1j}). \end{aligned}$$

So: $2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j=R^{-2}[4W_{12i}W_{12j}+Z_{1i}(Z_{12}Z_{2j}-Y_2Z_{1j})+Z_{2i}(Z_{12}Z_{1j}-Y_1Z_{2j})]$.

$i=1,\ j=2 \Rightarrow 2\mathrm{Tr}A_1A_2-\mathrm{Tr}A_1\mathrm{Tr}A_2=R^{-2}[Y_1(Z_{12}Y_2-Y_2Z_{12})+Z_{12}(Z_{12}^2-Y_1Y_2)] = Z_{12}$.

$i=1,\ j\geq 3 \Rightarrow 2\mathrm{Tr}A_1A_j-\mathrm{Tr}A_1\mathrm{Tr}A_j=R^{-2}[Y_1(Z_{12}Z_{2j}-Y_2Z_{1j})+Z_{12}(Z_{12}Z_{1j}-Y_1Z_{2j})] = Z_{1j}$.

$i=2,\ j\geq 3 \Rightarrow 2\mathrm{Tr}A_2A_j-\mathrm{Tr}A_2\mathrm{Tr}A_j=R^{-2}[Z_{12}(Z_{12}Z_{2j}-Y_2Z_{1j})+Y_2(Z_{12}Z_{1j}-Y_1Z_{2j})] = Z_{2j}$.

And finally if $3\leq i<j\leq n$: $2\mathrm{Tr}A_iA_j-\mathrm{Tr}A_i\mathrm{Tr}A_j=R^{-2}[\pi h(i,1,2,j)+R^2Z_{ij}] = Z_{ij}$.

(4) $$\begin{aligned} \mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j=\tfrac{1}{8}[&(a_i-d_i)(b_jc_k-c_jb_k)-(a_j-d_j)(b_ic_k-c_ib_k)+ \\ &+(a_k-d_k)(b_ic_j-c_ib_j)]. \end{aligned}$$

Since $b_ic_j-c_ib_j = 2R^{-1}(Z_{1i}Z_{2j}-Z_{2i}Z_{1j})$ we obtain:

$$2R^2(\mathrm{Tr}A_iA_jA_k-\mathrm{Tr}A_iA_kA_j) = -2W_{12i}(Z_{1j}Z_{2k}-Z_{2j}Z_{1k})+2W_{12j}(Z_{1i}Z_{2k}-Z_{2i}Z_{1k})+$$
$$-2W_{12k}(Z_{1i}Z_{2j}-Z_{2i}Z_{1j})$$
$$= \gamma(i,j,k).$$

We have to prove: $\gamma(i,j,k) = 2R^2W_{ijk}$ for all $1\leq i<j<k\leq n$.

$k\geq 3 \Rightarrow \gamma(1,2,k) = -2W_{12k}(Y_1Y_2-Z_{12}^2) = 2R^2W_{12k}.$

$$3\leq j<k \Rightarrow \gamma(1,j,k) = 2W_{12j}(Y_1Z_{2k}-Z_{12}Z_{1k})-2W_{12k}(Y_1Z_{2j}-Z_{12}Z_{1j}) =$$
$$= 2Y_1(Z_{2k}W_{12j}-Z_{2j}W_{12k})-2Z_{12}(Z_{1k}W_{12j}-Z_{1j}W_{12k}) =$$
$$= 2Y_1[-\pi g(2,1,j,k)-Y_2W_{1jk}+Z_{12}W_{2jk}]+$$
$$-2Z_{12}[\pi g(1,2,j,k)+Y_1W_{2jk}-Z_{12}W_{1jk}] =$$
$$= 2(Z_{12}^2-Y_1Y_2)W_{1jk} =$$
$$= 2R^2W_{1jk}.$$

$$3\leq j<k \Rightarrow \gamma(2,j,k) = 2W_{12j}(Z_{12}Z_{2k}-Y_2Z_{1k})-2W_{12k}(Z_{12}Z_{2j}-Y_2Z_{1j}) =$$
$$= 2Y_2(Z_{1j}W_{12k}-Z_{1k}W_{12j})+2Z_{12}(Z_{2k}W_{12j}-Z_{2j}W_{12k}) =$$
$$= 2Y_2[-\pi g(1,2,j,k)-Y_1W_{2jk}+Z_{12}W_{1jk}]+$$
$$+2Z_{12}[-\pi g(2,1,j,k)-Y_2W_{1jk}+Z_{12}W_{2jk}] =$$
$$= 2R^2W_{2jk}.$$

Finally, if $3\leq i<j<k\leq n$, we have:

$$\gamma(i,j,k) = -2W_{12i}(Z_{1j}Z_{2k}-Z_{2j}Z_{1k})+2W_{12j}(Z_{1i}Z_{2k}-Z_{2i}Z_{1k})-2W_{12k}(Z_{1i}Z_{2j}-Z_{2i}Z_{1j})=$$
$$= Z_{1i}(Z_{2k}W_{12j}-Z_{2j}W_{12k})+Z_{1j}(Z_{2i}W_{12k}-Z_{2k}W_{12i})+Z_{1k}(Z_{2j}W_{12i}-Z_{2i}W_{12j})+$$
$$+Z_{2i}(Z_{1j}W_{12k}-Z_{1k}W_{12j})+Z_{2j}(Z_{1k}W_{12i}-Z_{1i}W_{12k})+Z_{2k}(Z_{1i}W_{12j}-Z_{1j}W_{12i})=$$
$$= Z_{1i}[-\pi g(2,1,j,k)-Y_2W_{1jk}+Z_{12}W_{2jk}]+Z_{1j}[\pi g(2,1,i,k)+Y_2W_{1ik}-Z_{12}W_{2ik}]+$$
$$+Z_{1k}[-\pi g(2,1,i,j)-Y_2W_{1ij}+Z_{12}W_{2ij}]+Z_{2i}[-\pi g(1,2,j,k)-Y_1W_{2jk}+Z_{12}W_{1jk}]+$$
$$+Z_{2j}[\pi g(1,2,i,k)+Y_1W_{2ik}-Z_{12}W_{1ik}]+Z_{2k}[-\pi g(1,2,i,j)-Y_1W_{2ij}+Z_{12}W_{1ij}]=$$
$$= Y_1(-Z_{2i}W_{2jk}+Z_{2j}W_{2ik}-Z_{2k}W_{2ij})+Y_2(-Z_{1i}W_{1jk}+Z_{1j}W_{1ik}-Z_{1k}W_{1ij})+$$
$$+Z_{12}(Z_{1i}W_{2jk}-Z_{1j}W_{2ik}+Z_{1k}W_{2ij}+Z_{2i}W_{1jk}-Z_{2j}W_{1ik}+Z_{2k}W_{1ij}) =$$
$$= Y_1[-\pi g(2,i,j,k)-Y_2W_{ijk}]+Y_2[-\pi g(1,i,j,k)-Y_1W_{ijk}]+$$
$$+Z_{12}[\pi l(1,2,i,j,k)-\pi l(i,j,k,1,2)+2Z_{12}W_{ijk}]=$$
$$= 2R^2W_{ijk}.$$

<u>theorem 5.8</u>. There exists $\xi_0 = \mathrm{cl}(E^0;A_1^0,\dots,A_n^0) \in F(N_0)$ such that $\Phi(N_0)\xi_0 = p$. Moreover, (A_1^0,A_2^0) is irreducible. Hence $\xi_0 \in G(N_0)$.

<u>proof</u>: The automorphism of the Σ-algebra $\Sigma[R]$ such that $R \mapsto -R$ obviously induces an isomorphism $t : U_2 \xrightarrow{\sim} U_1$. Let $\xi_2 = \mathrm{cl}(\mathcal{O}_{U_2}^2;t^*A_1,\dots,t^*A_n) \in F(U_2)$ where the A_i are those of th. 5.7. Then $\Phi(U_2)\xi_2 = p|U_2$.

$U_1 \cap U_2 = N_0 \cap N_{(R-Z_{12})(R+Z_{12})+J_1} = N_0 \cap N_{Y_1Y_2+J_1} = N_0 \cap N_{Y_1+J_1} \cap N_{Y_2+J_1}$.

So $T = \begin{pmatrix} 0 & 1 \\ Y_1 & 0 \end{pmatrix}$ is an element of $Gl(2,\Gamma(U_1 \cap U_2,\mathcal{O}_{N_0}))$ and in $\Gamma(U_1 \cap U_2,\mathcal{O}_{N_0})$ we have:

$$TA_iT^{-1} = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ Y_1 & 0 \end{pmatrix}\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} 0 & Y_1^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d_i & Y_1^{-1}c_i \\ Y_1b_i & a_i \end{pmatrix} = \begin{pmatrix} t^*a_i & Y_1^{-1}c_i \\ Y_1b_i & t^*d_i \end{pmatrix}.$$

Now $Y_1b_i=Y_1R^{-1}(Z_{12}-R)^{-1}[Y_2Z_{1i}-Z_{2i}(Z_{12}-R)]=-R^{-1}[Y_1Z_{2i}-Z_{1i}(Z_{12}+R)]=t^*c_i$. Hence $TA_iT^{-1} = t^*A_i$ for all $1\le i\le n$.

Let E^0 be the locally free sheaf of $\mathcal{O}_{N_0}$-modules such that $E^0|U_1 \simeq \mathcal{O}_{U_1}^2$ and $E^0|U_2 \simeq \mathcal{O}_{U_2}^2$ where T is used as morphism of transition on $U_1 \cap U_2$. Then A_i and t^*A_i stick together and give an endomorphism A_i^0 of E^0 $(1\le i\le n)$. Define $\xi_0 = \mathrm{cl}(E^0;A_1^0,\dots,A_n^0) \in F(N_0)$. Then $\Phi(N_0)\xi_0 = p$.

Since $\mathrm{Tr}(A_1A_2)^2-\mathrm{Tr}A_1^2A_2^2 = \tfrac{1}{4}R^2 \neq 0$ in $\Gamma(U_1,\mathcal{O}_{U_1})$, the pair (A_1,A_2) is irreducible which implies the irreducibility of (A_1^0,A_2^0). Hence $\xi_0 \in G(N_0)$.

<u>theorem 5.9</u>. $\Phi_{ij} : \mathcal{Q}_{ij} \to h_{N_{ij}}$ is an epimorphism for all $1\le i<j\le n$, or equivalently: there exists $\xi_{ij} \in \mathcal{Q}_{ij}(N_{ij})$ such that $\Phi_{ij}(N_{ij})\xi_{ij} = id_{N_{ij}}$.

<u>proof</u>: Let $1\le p<q\le n$. Let $\sigma \in S_n$ be the permutation defined by $\sigma = (1p)(2q)$. We extend the k-algebra homomorphism $\mu_\sigma : \Sigma \to \Sigma$ (cf.IV.1.2) to $m'_\sigma : \Sigma[R] \to M$ such that $m'_\sigma|\Sigma = \mu_\sigma$ and $m'_\sigma(R) = R_{pq}$. Then we have

$m'_\sigma(J_1)=m'_\sigma[J\Sigma[R]+(R^2-Z_{12}^2+Y_1Y_2)] \subset JM+(R_{pq}^2-Z_{pq}^2+Y_pY_q) \subset J_0 \subset \sqrt{J_0}$ (cf.3.3 for the definition of J_0). Hence m'_σ induces a homomorphism $\overline{m}'_\sigma : \Sigma[R]/J_1 \to M/\sqrt{J_0}$. Since $\overline{m}'_\sigma(R+J_1) = R_{pq}+\sqrt{J_0}$ is invertible in $\left(M/\sqrt{J_0}\right)_{R_{pq}+\sqrt{J_0}}$, $\overline{m}'_\sigma$ induces a homomorphism $m_\sigma : \left(\Sigma[R]/J_1\right)_{R+J_1} \to \left(M/\sqrt{J_0}\right)_{R_{pq}+\sqrt{J_0}}$.

However $N_0 = \operatorname{Spec}\left(\Sigma[R]/J_1\right)_{R+J_1}$ and $N_{pq} = \operatorname{Spec}\left(M/\sqrt{J_0}\right)_{R_{pq}+\sqrt{J_0}}$. Therefore m_σ determines a morphism $f_\sigma : N_{pq} \to N_0$.

Let $E^0, A_1^0, \ldots, A_n^0$ be as in theorem 5.8. and define ξ_{pq} by

$$\xi_{pq} = cl(f_\sigma^* E^0; f_\sigma^* A_1^0, \ldots, f_\sigma^* A_n^0; \ldots R_{ij}+\sqrt{J_0} \ldots; \ldots R_{ijk}+\sqrt{J_0} \ldots).$$

I claim that ξ_{pq} belongs to $\mathcal{Q}_{pq}(N_{pq})$ and $\Phi_{pq}(N_{pq})\xi_{pq} = id_{N_{pq}}$.

Let $B_i = f_\sigma^* A^0_{\sigma^{-1}(i)}$. Then:

$$\begin{cases} TrB_i = m_\sigma TrA^0_{\sigma^{-1}(i)} = m_\sigma(X_{\sigma^{-1}(i)}+J_1) = X_i+\sqrt{J_0} & (1\le i\le n) \\ 2TrB_i^2-(TrB_i)^2 = m_\sigma(Y_{\sigma^{-1}(i)}+J_1) = Y_i+\sqrt{J_0} & (1\le i\le n) \\ 2TrB_iB_j-TrB_iTrB_j = m_\sigma[\pi Z_{\sigma^{-1}(i)\sigma^{-1}(j)}+J_1] = Z_{ij}+\sqrt{J_0} & (1\le i<j\le n) \\ TrB_iB_jB_k-TrB_iB_kB_j = m_\sigma[\pi W_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}+J_1] = W_{ijk}+\sqrt{J_0} & (1\le i<j<k\le n). \end{cases}$$

For all $1\le i<j\le n$, $R_{ij}+\sqrt{J_0} \in \Gamma(N_{pq},\mathcal{O}_{N_{pq}})$ and for all $1\le i<j<k\le n$ $R_{ijk}+\sqrt{J_0} \in \Gamma(N_{pq},\mathcal{O}_{N_{pq}})$. Their squares are equal to $4Tr(B_iB_j)^2-4TrB_i^2B_j^2$ and $4Tr[B_i(B_i+B_j+B_k)]^2-4TrB_i^2(B_i+B_j+B_k)^2$ respectively. Moreover, if x is a closed point of N_{pq} we have $4[Tr(B_pB_q)^2-TrB_p^2B_q^2](x)=[R_{pq}^2+\sqrt{J_0}](x)\neq 0$, so (B_p,B_q) is irreducible. Hence $\xi_{pq} \in \mathcal{Q}_{pq}(N_{pq})$.

Let $\beta = \Phi_{pq}(N_{pq})\xi_{pq} : N_{pq} \to N_{pq}$. Let β^* denote the corresponding endomorphism of $\left(M/\sqrt{J_0}\right)_{R_{pq}+\sqrt{J_0}}$ and let β_0 be the restriction of β^* to $M/\sqrt{J_0}$. Then by definition of Φ_{pq} we have:

$$\beta_0(X_i+\sqrt{J}_0) = \mathrm{Tr}B_i = X_i+\sqrt{J}_0$$
$$\beta_0(Y_i+\sqrt{J}_0) = 2\mathrm{Tr}B_i^2-(\mathrm{Tr}B_i)^2 = Y_i+\sqrt{J}_0$$
$$\beta_0(Z_{ij}+\sqrt{J}_0) = 2\mathrm{Tr}B_iB_j-\mathrm{Tr}B_i\mathrm{Tr}B_j = Z_{ij}+\sqrt{J}_0$$
$$\beta_0(W_{ijk}+\sqrt{J}_0) = \mathrm{Tr}B_iB_jB_k-\mathrm{Tr}B_iB_kB_j = W_{ijk}+\sqrt{J}_0$$

Hence β_0 is the inclusion map of $M/\sqrt{J}_0$ in $\left(M/\sqrt{J}_0\right)R_{pq}+\sqrt{J}_0$.
Therefore $\beta^* = \mathrm{id}$ and $\beta = \mathrm{id}_{N_{pq}}$.

<u>corollary 5.10</u>. $\Phi^a_{ij} : \mathcal{Q}^a_{ij} \to h_{N_{ij}}$ is an isomorphism for all $1\leq i<j\leq n$.
<u>proof</u>: In virtue of theorem 4.2 Φ^a_{ij} is a monomorphism. From theorem 5.9 above we know that Φ_{ij} is an epimorphism, and therefore also Φ^a_{ij}, because of the commutativity of

$$\begin{tikzcd} \mathcal{Q}^a_{ij} \arrow[r, "\Phi^a_{ij}"] & h_{N_{ij}} \\ \mathcal{Q}_{ij} \arrow[u] \arrow[ur, "\Phi_{ij}"'] & \end{tikzcd}$$

<u>remark</u>: For those who only see the celestial luminary in explicit formulas I write down ξ_{pq} explicitly.
$N_{pq} = U_1 \cup U_2$ where $U_1 = N_{pq}-V(R_{pq}-Z_{pq})$ and $U_2 = N_{pq}-V(R_{pq}+Z_{pq})$.
$\xi_{pq}|U_1 = \mathrm{cl}(\mathcal{O}^2_{U_1};A'_1,\ldots,A'_n;\ldots R_{ij}+\sqrt{J}_0\ldots;\ldots R_{ijk}+\sqrt{J}_0\ldots)$ where

$$\begin{cases} A'_p = \frac{1}{2}\begin{pmatrix} X_p & 1 \\ Y_p & X_p \end{pmatrix},\ A'_q = \frac{1}{2}\begin{pmatrix} X_p & Y_q(Z_{pq}-R_{pq})^{-1} \\ Z_{pq}-R_{pq} & X_q \end{pmatrix} \text{ and if } i \notin \{p,q\} \\ \\ A'_i = \frac{1}{2}\begin{pmatrix} X_i-2R_{pq}^{-1}\pi(W_{pqi}) & R_{pq}^{-1}(Z_{pq}-R_{pq})^{-1}[Y_q\pi(Z_{pi})-\pi(Z_{qi})(Z_{pq}-R_{pq})] \\ R_{pq}^{-1}[Y_p\pi(Z_{qi})-\pi(Z_{pi})(Z_{pq}-R_{pq})] & X_i+2R_{pq}^{-1}\pi(W_{pqi}) \end{pmatrix} \end{cases}$$

$\xi_{pq}|U_2 = \mathrm{cl}(\mathcal{O}^2_{U_2};B'_1,\ldots,B'_n;\ldots R_{ij}+\sqrt{J}_0\ldots;\ldots R_{ijk}+\sqrt{J}_0\ldots)$, where the B'_i arise from the A'_i by changing R_{pq} into $-R_{pq}$.

<u>theorem 5.11</u>. $\Phi_{ijk} : \mathcal{Q}_{ijk} \to h_{N_{ijk}}$ is an epimorphism for all $1 \le i<j<k \le n$.

<u>proof</u>: Let $1 \le p<q<r \le n$ and define $\sigma \in S_n$ by $\sigma = (3r)(2q)(1p)$. Then $\sigma^{-1}(1) = p$, $\sigma^{-1}(2) = q$ and $\sigma^{-1}(3) = r$ whatever p,q and r may be. Let $\Delta_0' : \Sigma[R] \to M$ be the k-algebra homomorphism such that $\Delta_0'(R) = R_{pqr}$ and $\Delta_0'|\Sigma = \mu_{\sigma^{-1}}\mu_{(123)}\delta\mu_{(132)}$ (cf. IV.1.2 and IV.1.5.2).

Since $\delta(J) = J$ and $\mu_\sigma J = J$ (cf. IV.2.17 and IV.2.7) we have

$$\Delta_0'(J\Sigma[R]) \subset JM \subset \sqrt{J_0}$$

$$\begin{aligned}\Delta_0'(R^2-Z_{12}^2+Y_1Y_2) &= R_{pqr}^2-\mu_{\sigma^{-1}}\mu_{(123)}\delta\mu_{(132)}(Z_{12}^2-Y_1Y_2).\\ &= R_{pqr}^2-\mu_{\sigma^{-1}}[Z_{12}^2-Y_1Y_2+Z_{13}^2-Y_1Y_3+2(Z_{12}Z_{13}-Y_1Z_{23})]\\ &= R_{pqr}^2-[Z_{pq}^2-Y_pY_q+Z_{pr}^2-Y_pY_r+2(Z_{pq}Z_{pr}-Y_pZ_{qr})] \in J_0 \subset \sqrt{J_0}.\end{aligned}$$

Hence Δ_0' induces a homomorphism $\Delta_0 : \left(\Sigma[R]/J_1\right)_{R+J_1} \to \left(M/\sqrt{J_0}\right)_{R_{pqr}+\sqrt{J_0}}$ and Δ_0 determines a morphism $\Delta : N_{pqr} \to N_0$.

Define $\xi_{pqr} = \mathrm{cl}(\Delta^*E^0;B_1,\dots,B_n;\dots R_{ij}+\sqrt{J_0}\dots;\dots R_{ijk}+\sqrt{J_0}\dots)$ where

$$\begin{cases} B_q = \Delta^*(A^0_{\sigma(q)}+A^0_{\sigma(r)}) = \Delta^*(A_2^0+A_3^0)\\ B_r = -\Delta^*A^0_{\sigma(r)} = -\Delta^*A_3^0\\ B_i = \Delta^*A^0_{\sigma(i)} \text{ if } i \notin \{p,q\}\end{cases}$$

I claim: $\xi_{pqr} \in \mathcal{Q}_{pqr}(N_{pqr})$ and $\Phi(N_{pqr})\xi_{pqr} = \mathrm{id}$. Looking back at the proof of theorem 5.9 it is clear that we only have to show that $(B_1,\dots,B_n)$ has the "good invariants", i.e., $\mathrm{Tr}B_i = X_i+\sqrt{J_0}$ etc.

(a) $\Delta_0'(X_1) = X_p$, $\Delta_0'(X_2) = X_q+X_r$, $\Delta_0'(X_3) = -X_r$ and $\Delta_0'(X_i) = \mu_{\sigma^{-1}}(X_i)$ if $i \ge 4$.

Hence $\mathrm{Tr}B_p = \mathrm{Tr}\Delta^*A^0_{\sigma(p)} = \mathrm{Tr}\Delta^*A_1^0 = \Delta_0(X_1+J_1) = \Delta_0'X_1+\sqrt{J_0} = X_p+\sqrt{J_0}$

$\mathrm{Tr}B_q = \mathrm{Tr}\Delta^*A_2^0+\mathrm{Tr}\Delta^*A_3^0 = \Delta_0(X_2+X_3+J_1) = X_q+\sqrt{J_0}$

$\mathrm{Tr}B_r = \mathrm{Tr}\Delta^*A_3^0 = -\Delta_0(X_3+J_1) = X_r+\sqrt{J_0}$

$i \notin \{p,q,r\}$, or $\sigma(i) \ge 4 \Rightarrow \mathrm{Tr}B_i=\mathrm{Tr}\Delta^*A^0_{\sigma(i)}=\mu_{\sigma^{-1}}X_{\sigma(i)}+\sqrt{J_0}=X_i+\sqrt{J_0}$.

(b)$\Delta_0'(Y_1) = Y_p$, $\Delta_0'(Y_2) = Y_q+Y_r+2Z_{qr}$, $\Delta_0'(Y_3) = Y_r$ and $\Delta_0'(Y_i) = \mu_{\sigma^{-1}}(Y_i)$ if $i \geq 4$.

Hence $2\mathrm{Tr}B_p^2-(\mathrm{Tr}B_p)^2 = \Delta_0[2\mathrm{Tr}(A_1^0)^2-(\mathrm{Tr}A_1^0)^2] = \Delta_0(Y_1+J_1) = Y_p+\sqrt{J_0}$

$$2\mathrm{Tr}B_q^2-(\mathrm{Tr}B_q)^2 = \Delta_0(Y_2+Y_3+2Z_{23}+J_1) = Y_q+\sqrt{J_0}$$

$$2\mathrm{Tr}B_r^2-(\mathrm{Tr}B_r)^2 = \Delta_0(Y_3+J_1) = Y_r+\sqrt{J_0}$$

$$i \notin \{p,q,r\} \Rightarrow 2\mathrm{Tr}B_i^2-(\mathrm{Tr}B_i)^2 = \Delta_0(Y_{\sigma(i)}+J_1) = Y_i+\sqrt{J_0}.$$

(c) $\Delta_0'(Z_{12}) = Z_{pq}+Z_{pr}$, $\Delta_0'(Z_{13}) = -Z_{pr}$, $\Delta_0'(Z_{1j}) = \mu_{\sigma^{-1}}(Z_{ij})$ if $j \geq 4$, $\Delta_0'(Z_{23}) = -Z_{qr}-Y_r$, $\Delta_0'(Z_{2j}) = \mu_{\sigma^{-1}}(Z_{2j}+Z_{3j})$ if $j \geq 4$, $\Delta_0'(Z_{3j}) = -\mu_{\sigma^{-1}}(Z_{3j})$ if $j \geq 4$ and $\Delta_0'(Z_{ij}) = \mu_{\sigma^{-1}}(Z_{ij})$ if $4 \leq i < j \leq n$.

If $\{i,j\} \cap \{p,q,r\} = \emptyset$ then $\{\sigma(i),\sigma(j)\} \cap \{1,2,3\} = \emptyset$ and

$$\begin{aligned}
2\mathrm{Tr}B_iB_j-\mathrm{Tr}B_i\mathrm{Tr}B_j &= \Delta_0[2\mathrm{Tr}A^0_{\sigma(i)}A^0_{\sigma(j)}-\mathrm{Tr}A^0_{\sigma(i)}\mathrm{Tr}A^0_{\sigma(j)}] = \\
&= \Delta_0[\mu_\sigma(Z_{ij})+J_1] = \\
&= Z_{ij}+\sqrt{J_0}.
\end{aligned}$$

$$\begin{aligned}
2\mathrm{Tr}B_pB_q-\mathrm{Tr}B_p\mathrm{Tr}B_q &= \Delta_0[2\mathrm{Tr}A_1^0(A_2^0+A_3^0)-\mathrm{Tr}A_1^0\mathrm{Tr}(A_2^0+A_3^0)] = \\
&= \Delta_0(Z_{12}+Z_{13}+J_1) = Z_{pr}+\sqrt{J_0}.
\end{aligned}$$

$$2\mathrm{Tr}B_pB_r-\mathrm{Tr}B_p\mathrm{Tr}B_r = -\Delta_0[2\mathrm{Tr}A_1^0A_3^0-\mathrm{Tr}A_1^0\mathrm{Tr}A_3^0] = -\Delta_0(Z_{13}+J_1) = Z_{pr}+\sqrt{J_0}.$$

$$\begin{aligned}
2\mathrm{Tr}B_qB_r-\mathrm{Tr}B_q\mathrm{Tr}B_r &= -\Delta_0[2\mathrm{Tr}(A_2^0+A_3^0)A_3^0-\mathrm{Tr}(A_2^0+A_3^0)\mathrm{Tr}A_3^0] = \\
&= -\Delta_0(Z_{23}+Y_3+J_1) = Z_{qr}+\sqrt{J_0}.
\end{aligned}$$

$\{i,j\} \cap \{p,q,r\} = \{p\} \Rightarrow 2\mathrm{Tr}B_iB_j-\mathrm{Tr}B_i\mathrm{Tr}B_j = \Delta_0(\mu_\sigma Z_{ij}+J_1) = Z_{ij}+\sqrt{J_0}.$

$\{i,j\} \cap \{p,q,r\} = \{r\} \Rightarrow 2\mathrm{Tr}B_iB_j-\mathrm{Tr}B_i\mathrm{Tr}B_j = \Delta_0(-\mu_\sigma Z_{ij}+J_1) = Z_{ij}+\sqrt{J_0}.$

If $i \notin \{p,q,r\}$, so $\sigma(i) \geq 4$, we finally have:

$$\begin{aligned}
2\mathrm{Tr}B_iB_q-\mathrm{Tr}B_i\mathrm{Tr}B_q &= \Delta_0[2\mathrm{Tr}A^0_{\sigma(i)}(A_2^0+A_3^0)-\mathrm{Tr}A^0_{\sigma(i)}\mathrm{Tr}(A_2^0+A_3^0)] = \\
&= \Delta_0(Z_{2,\sigma(i)}+Z_{3,\sigma(i)}+J_1) = \mu_{\sigma^{-1}}Z_{2,\sigma(i)}+\sqrt{J_0} = \pi Z_{iq}+\sqrt{J_0}.
\end{aligned}$$

(d) $\Delta_0'(W_{123}) = -W_{pqr}$

$\Delta_0'(W_{12k}) = \mu_{\sigma^{-1}}(W_{12k}+W_{13k})$ if $k \geq 4$

$\Delta_0'(W_{13k}) = -\mu_{\sigma^{-1}}(W_{13k})$

$\Delta_0'(W_{1jk}) = \mu_{\sigma^{-1}}(W_{1jk})$ if $4 \leq j < k \leq n$

$\Delta_0'(W_{23k}) = -\mu_{\sigma^{-1}}(W_{23k})$

$\Delta_0'(W_{2jk}) = \mu_{\sigma^{-1}}(W_{2jk}+W_{3jk})$ if $4 \leq j < k \leq n$

$\Delta_0'(W_{3jk}) = -\mu_{\sigma^{-1}}(W_{3jk})$

$\Delta_0'(W_{ijk}) = \mu_{\sigma^{-1}}(W_{ijk})$ if $4 \leq i < j < k \leq n$.

If $\{i,j,k\} \cap \{p,q,r\} = \emptyset$ then obviously $TrB_iB_jB_k-TrB_iB_kB_j = W_{ijk}+\sqrt{J}_0$.

If $\{j,k\} \cap \{p,q,r\} = \emptyset$ and $j \neq k$, then:

$$\begin{aligned} TrB_pB_jB_k-TrB_pB_kB_j &= \Delta_0[TrA_1^0A_{\sigma(j)}^0A_{\sigma(k)}^0-TrA_1^0A_{\sigma(k)}^0A_{\sigma(j)}^0] = \\ &= \Delta_0[\pi W_{1,\sigma(j),\sigma(k)}+J_1] = \\ &= \mu_{\sigma^{-1}}\pi W_{1,\sigma(j),\sigma(k)}+\sqrt{J}_0 \\ &= \pi W_{pjk}+\sqrt{J}_0. \end{aligned}$$

Hence $TrB_iB_jB_k-TrB_iB_kB_j = W_{ijk}+\sqrt{J}_0$ if $i<j<k$ and $\{i,j,k\} \cap \{p,q,r\} = \{p\}$.

If $\{j,k\} \cap \{p,q,r\} = \emptyset$ and $j \neq k$, then:

$$\begin{aligned} TrB_qB_jB_k-TrB_qB_kB_j &= \Delta_0[Tr(A_2^0+A_3^0)A_{\sigma(j)}^0A_{\sigma(k)}^0-Tr(A_2^0+A_3^0)A_{\sigma(k)}^0A_{\sigma(j)}^0] = \\ &= \Delta_0[\pi W_{2,\sigma(j),\sigma(k)}+\pi W_{3,\sigma(j),\sigma(k)}+J_1] = \\ &= \mu_{\sigma^{-1}}\ W_{2,\sigma(j),\sigma(k)}+\sqrt{J}_0 = \\ &= \pi W_{qjk}+\sqrt{J}_0. \end{aligned}$$

Hence $TrB_iB_jB_k-TrB_iB_kB_j = W_{ijk}+\sqrt{J}_0$ if $i<j<k$ and $\{i,j,k\} \cap \{p,q,r\} = \{q\}$.

If $\{j,k\} \cap \{p,q,r\} = \emptyset$ and $j \neq k$, then:

$$\begin{aligned} TrB_rB_jB_k-TrB_rB_kB_j &= \Delta_0[-TrA_3^0A_{\sigma(j)}^0A_{\sigma(k)}^0+TrA_3^0A_{\sigma(k)}^0A_{\sigma(j)}^0] = \\ &= \Delta_0(-\pi W_{3,\sigma(j),\sigma(k)}+J_1) = \\ &= \pi W_{rjk}+\sqrt{J}_0. \end{aligned}$$

Hence $TrB_iB_jB_k-TrB_iB_kB_j = W_{ijk}+\sqrt{J}_0$ if $i<j<k$ and $\{i,j,k\} \cap \{p,q,r\} = \{r\}$.

If $i \notin \{p,q,r\}$ then:

$$\mathrm{Tr}B_pB_qB_i - \mathrm{Tr}B_pB_iB_q = \Delta_0[\mathrm{Tr}A_1^0(A_2^0+A_3^0)A_{\sigma(i)}^0 - \mathrm{Tr}A_1^0A_{\sigma(i)}^0(A_2^0+A_3^0)] =$$
$$= \Delta_0(W_{1,2,\sigma(i)}+W_{1,3,\sigma(i)}+J_1) = \pi W_{pqi}+\sqrt{J_0},$$

$$\mathrm{Tr}B_pB_rB_i - \mathrm{Tr}B_pB_iB_r = -\Delta_0[\mathrm{Tr}A_1^0A_3^0A_{\sigma(i)}^0 - \mathrm{Tr}A_1^0A_3^0A_{\sigma(i)}^0] =$$
$$= -\Delta_0(W_{1,3,\sigma(i)}+J_1) = \pi W_{pri}+\sqrt{J_0}, \text{ and}$$

$$\mathrm{Tr}B_qB_rB_i - \mathrm{Tr}B_qB_iB_r = -\Delta_0[\mathrm{Tr}(A_2^0+A_3^0)A_3^0A_{\sigma(i)}^0 - \mathrm{Tr}(A_2^0+A_3^0)A_{\sigma(i)}^0A_3^0] =$$
$$= -\Delta_0(W_{2,3,\sigma(i)}+J_1) = \pi W_{qri}+\sqrt{J_0}.$$

Therefore $\mathrm{Tr}B_iB_jB_k - \mathrm{Tr}B_iB_kB_j = W_{ijk}+\sqrt{J_0}$ if $i<j<k$ and $\#\{i,j,k\} \cap \{p,q,r\}=2$.

Finally we have: $\mathrm{Tr}B_pB_qB_r - \mathrm{Tr}B_pB_rB_q = -\Delta_0[\mathrm{Tr}A_1^0(A_2^0+A_3^0)A_3^0 - \mathrm{Tr}A_1^0A_3^0(A_2^0+A_3^0)] =$

$$= -\Delta_0[\mathrm{Tr}A_1^0A_2^0A_3^0 - \mathrm{Tr}A_1^0A_3^0A_2^0] =$$
$$= -\Delta_0(W_{123}+J_1) = W_{pqr}+\sqrt{J_0}.$$

<u>corollary 5.12</u>. $\Phi_{ijk}^a : \mathcal{Q}_{ijk}^a \to h_{N_{ijk}}$ ia sn isomorphism for all $1 \leq i<j<k \leq n$.

<u>proof</u>: Evident in view of 5.11 and 4.3.

<u>Combining cor. 5.12, cor. 5.10 and theorem 3.18 we can state the desired result, namely $\Phi^a : P^a \to h_M$ is an isomorphism.</u>

For the sake of completeness, I give $\xi_{pqr} \in \mathcal{Q}_{pqr}(N_{pqr})$ explicitly.

$N_{pqr} = U_1 \cup U_2$ where $U_1 = N_{pqr} - V(R_{pqr}-Z_{pq}-Z_{pr})$ and

$U_2 = N_{pqr} - V(R_{pqr}+Z_{pq}+Z_{pr})$.

$\xi_{pqr}|U_1 = \mathrm{cl}(\mathcal{O}_{U_1}^2; A_1'', \ldots, A_n''; \ldots R_{ij}+\sqrt{J_0}\ldots; \ldots R_{ijk}+\sqrt{J_0}\ldots)$ where,

abbreviating $Z_{pq}+Z_{pr}-R_{pqr}$ to Ω:

$$A_p'' = \tfrac{1}{2}\begin{pmatrix} X_p & 1 \\ Y_p & X_p \end{pmatrix},$$

$$A_q'' = \tfrac{1}{2}\begin{pmatrix} X_q+2R_{pqr}^{-1}W_{pqr} & R_{pqr}^{-1}\Omega^{-1}[(Y_q+Y_r+2Z_{qr})Z_{pq}-\Omega(Y_q+Z_{qr})] \\ R_{pqr}^{-1}[Y_p(Y_q+Z_{qr})-\Omega Z_{pq}] & X_q-2R_{pqr}^{-1}W_{pqr} \end{pmatrix}$$

$$A_r'' = \tfrac{1}{2}\begin{pmatrix} X_r - 2R_{pqr}^{-1}W_{pqr} & R_{pqr}^{-1}\Omega^{-1}[(Y_q+Y_r+2Z_{qr})Z_{pr}-\Omega(Y_r+Z_{qr})] \\ R_{pqr}^{-1}[Y_p(Y_r+Z_{qr})-\Omega Z_{pr}] & X_r+2R_{pqr}^{-1}W_{pqr} \end{pmatrix},$$

and if $i \notin \{p,q,r\}$:

$$A_i'' = \tfrac{1}{2}\begin{pmatrix} X_i - 2R_{pqr}^{-1}\pi(W_{pqi}+W_{pri}) & R_{pqr}^{-1}\Omega^{-1}[(Y_q+Y_r+2Z_{qr})\pi Z_{pi}-\Omega\pi(Z_{qi}+Z_{ri})] \\ R_{pqr}^{-1}[Y_p\pi(Z_{qi}+Z_{ri})-\Omega\pi Z_{pi}] & X_i+2R_{pqr}^{-1}\pi(W_{pqi}+W_{pri}) \end{pmatrix}.$$

$\xi_{pqr}|U_2 = \mathrm{cl}(\mathcal{O}^2_{U_2};B_1'',\ldots,B_n'';\ldots R_{ij}+\sqrt{J_0}\ldots;\ldots R_{ijk}+\sqrt{J_0}\ldots)$, the B_i'' arising from the A_i'' by changing R_{pqr} into $-R_{pqr}$.

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INDEX OF SYMBOLS

$M(2,\Omega)$: the 2×2 matrices with entries in the set Ω.

id_V: identity mapping on the set V.

$\#$: cardinality.

iff: if and only if.

S_n : the group of permutations of $\{1,\ldots,n\}$.

SAMENVATTING

Laat k een algebraïsch afgesloten lichaam van karakteristiek nul zijn. Zij $M(2,k)^n$ de verzameling der geordende n-tallen 2×2 matrices met coëfficiënten in k. We beschouwen de volgende equivalentierelatie op $M(2,k)^n$: $(A_1,\dots,A_n) \sim (B_1,\dots,B_n)$ dan en slechts dan als er een element T van $Gl(2,k)$ bestaat zo dat $B_i = T^{-1}A_iT$ voor alle $1 \leq i \leq n$. We beperken ons tot de gevallen $n \geq 2$. (Het geval n = 2 is ook bestudeerd door W. Dekkers [2]).

Zij $R = k[\underline{X}_1,\dots,\underline{X}_n]$ de veeltermring in 4n onbepaalden waarbij $\underline{X}_i$ staat voor de vier onbepaalden $X_{i;11}$, $X_{i;12}$, $X_{i;21}$ en $X_{i;22}$. De elementen van $M(2,k)^n$ corresponderen met de gesloten punten van de varieteit X = Spec R. Zij $S = k[T_{11},T_{12},T_{21},T_{22},(T_{11}T_{22}-T_{12}T_{21})^{-1}]$ en G = Spec S. Dan kan de bovenstaande equivalentierelatie vertaald worden in een algebraïsche werking van de affiene algebraïsche groep G op de varieteit X d.m.v. een k-algebra homomorfisme $\sigma^* : R \to S \otimes_k R$.

In het eerste deel van dit proefschrift (i.e. hoofdstuk I tot en met V) wordt onderzocht welke invarianten men kan hechten aan de equivalentieklassen van de elementen van $M(2,k)^n$ onder de vernoemde equivalentierelatie, wat de algebraïsche relaties tussen deze invarianten zijn en in hoeverre zij die equivalentieklassen bepalen. Tegelijkertijd bestuderen we het algebraïsch-meetkundige equivalent van deze equivalentierelatie, te weten, de werking van G op X.

De elementen van R kunnen opgevat worden als veeltermfunkties op $M(2,k)^n$ met waarden in k. De subalgebra $R^G = \{x \in R \mid \sigma^*(x) = 1 \otimes x\}$ correspondeert daarbij met de funkties die constant zijn op de equivalentieklassen. De 2×2 matrix $\begin{pmatrix} X_{i;11} & X_{i;12} \\ X_{i;21} & X_{i;22} \end{pmatrix}$ met coëfficiënten in R noterend met $\underline{X}_i$, leiden we uit het werk van C. Procesi [11] af dat R^G als k-algebra wordt voortgebracht door $Tr\underline{X}_i$ $(1 \leq i \leq n)$, $Tr\underline{X}_i^2$ $(1 \leq i \leq n)$, $Tr\underline{X}_i\underline{X}_j$ $(1 \leq i < j \leq n)$ en $Tr\underline{X}_i\underline{X}_j\underline{X}_k$ $(1 \leq i < j < k \leq n)$.

Enigszins anders geformuleerd: Zij Σ de veeltermring over k in de $m = 2n+\binom{n}{2}+\binom{n}{3}$ onbepaalden X_i $(1 \leq i \leq n)$, Y_i $(1 \leq i \leq n)$, Z_{ij} $(1 \leq i < j \leq n)$ en W_{ijk} $(1 \leq i < j < k \leq n)$ en zij $\alpha : \Sigma \to R$ het k-algebra homomorfisme zó dat $\alpha(X_i) = Tr\underline{X}_i$, $\alpha(Y_i) = 2Tr\underline{X}_i^2-(Tr\underline{X}_i)^2$, $\alpha(Z_{ij}) = 2Tr\underline{X}_i\underline{X}_j-Tr\underline{X}_iTr\underline{X}_j$ en $\alpha(W_{ijk}) = Tr\underline{X}_i\underline{X}_j\underline{X}_k-Tr\underline{X}_i\underline{X}_k\underline{X}_j$. Dan geldt $\alpha(\Sigma) = R^G$ en dus $R^G \simeq \Sigma/\ker \alpha$.

De invarianten kennen we nu. Ze zijn verwerkt in de afbeelding $i : M(2,k)^n \to k^m$ die aan $(A_1,\ldots,A_n) \in M(2,k)^n$ toevoegt:

$$(..TrA_i..,..2TrA_i^2-(TrA_i)^2..,..2TrA_iA_j-TrA_iTrA_j..,..TrA_iA_jA_k-TrA_iA_kA_j..).$$

Het bepalen van de algebraïsche relaties tussen deze invarianten komt overeen met het zoeken naar een eindig aantal voortbrengers van ker α. Deze voortbrengers worden expliciet gegeven (C. Procesi geeft in [11] ook informatie over deze algebraïsche relaties maar komt daarbij niet tot een eindig aantal voortbrengende relaties). Dit is een van de hoofdresultaten van het proefschrift. Het bewijs dat ker α inderdaad wordt voortgebracht door de elementen van Σ die we in definitie 1.8 van hoofdstuk IV geven is nogal gecompliceerd.

i is ten duidelijkste constant op de equivalentieklassen, maar "onderscheidt" ze in het algemeen niet. Zo zijn $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ en $(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ niet equivalent maar ze hebben hetzelfde beeld onder $i : M(2,k)^2 \to k^5$. Er wordt aangetoond dat die "scheiding" van equivalentieklassen wel optreedt als we i beperken tot de irreducibele elementen van $M(2,k)^n$; dat zijn de elementen die <u>niet</u> equivalent zijn met een element van de vorm $(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},\ldots,\begin{pmatrix} * & * \\ 0 & * \end{pmatrix})$.

De werking van G op X wordt in algemenere situaties bestudeerd in de quotiëntentheorie (door o.a. D. Mumford in [9] en [10]). Uit deze algemenere theorie volgt: Zij $Y = \operatorname{Spec} R^G$, $\phi : X \to Y$ het morfisme geïnduceerd door de inclusie $R^G \subset R$ en X_S het open invariante deel van X zó dat een gesloten punt x van X tot X_S behoort dan en slechts dan als de baan van x (onder G) gesloten is en de stabilisator van x minimale dimensie heeft. Zij tenslotte $Y_S = \phi(X_S)$ en $\phi'\ X_S \to Y_S$ de restrictie van $\phi : X \to Y$. Dan is het paar (Y,ϕ) een quotiënt voor de werking van G op X en het paar (Y_S,ϕ') een geometrisch quotiënt voor de werking van G op X_S (zie §2 van hoofdstuk I voor de definitie van het begrip (geometrisch) quotiënt).

We bewijzen dat de gesloten punten van X_S corresponderen met de irreducibele elementen van $M(2,k)^n$. Uit een nadere bestudering van Y_S en Y blijkt dat Y_S, in de gevallen $n \geq 3$, het gladde deel van Y is. Dit is een ander hoofdresultaat van dit proefschrift waarbij we onze kennis omtrent de voortbrengers van ker α benutten (zie hoofdstuk IV, §3).

Het niet-gladde deel $Y-Y_S$ van Y blijkt isomorf te zijn met $\mathbb{A}^n * \mathbb{A}^n$, het symmetrisch produkt van $\mathbb{A}^n$ met zichzelf. Vervolgens onderzoeken we dan nog de structuur van de coördinaatring van $\mathbb{A}^n * \mathbb{A}^n$ en bewijzen dat het niet-gladde deel van $\mathbb{A}^n * \mathbb{A}^n$, en dus ook van $Y-Y_S$, isomorf is met $\mathbb{A}^n$.

In het tweede deel (hoofdstuk VI) wordt ingegaan op het probleem van de existentie van grove- of fijne moduliruimten voor de functoren van de categorie der varieteiten naar de categorie der verzamelingen die op natuurlijke wijze samenhangen met het in het eerste deel gestelde probleem.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 1 oktober 1948 te Geleen geboren. In 1966 behaalde hij aan het Bisschoppelijk College te Sittard het diploma gymnasium β. Vervolgens studeerde hij wis- en natuurkunde aan de Nijmeegse Universiteit. Hij volgde colleges bij o.a. de hoogleraren dr. J.H. de Boer, drs. J.J. de Iongh, dr. A.H.M. Levelt, dr. A.C.M. van Rooij, dr. H.A.M.J. Oedayrajsingh Varma en dr. H. de Vries.

Op 7 oktober 1971 legde hij cum laude het doctoraalexamen wiskunde af met als specialisatie algebraïsche topologie, waarin hij onderwezen was door prof.dr. H.A.M.J. Oedayrajsingh Varma. Vanaf die datum was hij als wetenschappelijk medewerker verbonden aan het Mathematisch Instituut van de Nijmeegse Universiteit. In deze periode heeft hij onder leiding van prof.dr. A.H.M. Levelt onderzoek verricht op het terrein van de algebraïsche meetkunde. Dit onderzoek heeft de grondslag gevormd voor dit proefschrift.

Vanaf 1 augustus 1976 is hij als leraar wiskunde verbonden aan het Stedelijk Lyceum te Maastricht.

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STELLINGEN

1. Een partitie van een natuurlijk getal n is een representatie van n als som van een aantal positieve gehele getallen waarbij de volgorde der getallen in deze som irrelevant geacht wordt (cf. [*]). P(n) zij de verzameling der partities van n. De orde van $a_1+a_2+\ldots+a_k \in P(n)$ is per definitie $\mathrm{kgv}(a_1,a_2,\ldots,a_k)$ en $\alpha(n)$ is het maximum van de orden van de elementen van P(n).

Dan geldt:

(1) $\alpha(n) = n \Leftrightarrow n \in \{1,2,3,4,6\}$.

(2) $\alpha(n) = \alpha(n-1) = n \Leftrightarrow n = 6$. De elementen van orde $6 = \alpha(6)$ in P(6) zijn 6 en 1+2+3.

Zij $p_1^{\alpha_1}\ldots p_r^{\alpha_r}$ een priemdecompositie van $\alpha(n)$ met $p_i \neq p_j$ als $i \neq j$. Dan geldt bovendien:

(3) Als $\alpha(n) > \alpha(n-1)$ dan is $p_1^{\alpha_1}+\ldots+p_r^{\alpha_r}$ het enige element in P(n) van orde $\alpha(n)$. In het bijzonder: $n = \sum_{i=1}^{r} p_i^{\alpha_i}$.

(4) Als $\alpha(n) = \alpha(n-1) > \alpha(n-2)$ dan is $1+p_1^{\alpha_1}+\ldots+p_r^{\alpha_r}$ het enige element in P(n) van orde $\alpha(n)$, tenzij n = 6 (zie (2)).

(5) Als $\alpha(n) = \alpha(n-1) =\ldots= \alpha(n-s) > \alpha(n-s-1)$ voor zekere $s \geq 2$ ($\Rightarrow n \geq 21$) dan zijn $1,2,\ldots,s$ delers van $\alpha(n)$ en als $a_1+\ldots+a_k \in P(n)$ orde $\alpha(n)$ heeft dan is voldaan aan:

(i) $k \neq 1$ en $\mathrm{ggd}(a_i,a_j) \leq s$ voor alle $i \neq j$.

(ii) als $\mathrm{ggd}(a_i,a_j) = 1$ voor alle $i \neq j$ dan is $a_1+\ldots+a_k$ gelijk aan de partitie $1+1+\ldots+1+p_1^{\alpha_1}+\ldots+p_r^{\alpha_r}$ (waarin s enen optreden).

(iii) als $\mathrm{ggd}(a_i, a_j) = s$ voor zekere $i \neq j$ dan is $a_1+\ldots+a_k$ gelijk aan de partitie $s+p_1^{\alpha_1}+\ldots+p_r^{\alpha_r}$. In geval $s \geq 3$ geldt bovendien: $s = p_t^{\alpha_t}$ voor zekere $1 \leq t \leq r$.

[*] G.H. Hardy and E.M. Wright: "An introduction to the theory of numbers"; Oxford, At the Clarendon Press 1954, p.273.

2. Zij S_n de groep der permutaties van $\{1,\ldots,n\}$. Voor elke $0 \leq p \leq n$ zij $V(p)$ de deelverzameling van S_n bestaande uit de permutaties die precies p verschillende elementen van $\{1,\ldots,n\}$ invariant laten. Dan geldt: $\sum_{\sigma \in V(p)} \mathrm{sgn}\,\sigma = \binom{n}{n-p}(n-p-1)(-1)^{n-p-1}$.

3. Zij $\mathcal{C}$ een categorie en Ens de categorie der verzamelingen. Laat $\mathcal{F}$ en $\mathcal{G}$ contravariante functoren van $\mathcal{C}$ naar Ens zijn en $u : \mathcal{F} \to \mathcal{G}$ een functormorfisme. Uitgaande van de veronderstelling dat de functor $\mathcal{G}$ representeerbaar is, geeft A. Grothendieck in [*] een criterium opdat $\mathcal{F}$ representeerbaar is. Dat criterium kan als volgt verscherpt worden:

Als $\mathcal{G}$ representeerbaar is door $\langle Y;\eta\rangle$ en $\mathcal{F}_\eta : \mathcal{C}/Y \to \mathrm{Ens}$ door $\langle (X,f);\xi\rangle$ dan is $\mathcal{F}$ representeerbaar door $\langle X;\xi\rangle$.

[*] Séminaire Cartan, 13[e] année 1960/61, fasc. I, exp. 11, lemme 3.6 en proposition 3.7.

4. Zij k een lichaam van karakteristiek nul en $M(n,k)$ de verzameling der $n\times n$ matrices met coëfficiënten in k. $(A_1,\ldots,A_k) \in M(n,k)^m$ heet reducibel als de kanoniek met de A_i corresponderende lineaire transformaties van k^n een gemeenschappelijke niet-triviale invariante

lineaire deelruimte bezitten. Irreducibel = niet reducibel. $(A_1,\ldots,A_m)$ heet nilpotent als er een $T \in Gl(n,k)$ bestaat zó dat $T^{-1}A_iT$ voor elke $1 \leq i \leq m$ van nevenstaande vorm is.

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ \theta & & 0 \end{pmatrix}$$

De reductieve groep $G = PGl(n,k)$ werkt op $M(n,k)^m$ door gelijktijdige conjugatie. Zij $X = \mathbb{P}(M(n,k)^m) = \mathbb{P}^{mn^2-1}$ en $L = \mathcal{O}_X(1)$, een ampele schoof op X waarnaar de werking van G op X gelift kan worden (cf. [*], Ch.I, §3).

Gebruikmakend van het numerieke criterium voor (semi)stabiliteit ([*], Ch.II, §1) kan men dan aantonen:

Als het gesloten punt $x \in X$ correspondeert met $A = (A_1,\ldots,A_m)$ dan geldt:

(a) x stabiel $\Leftrightarrow$ A irreducibel.

(b) x semistabiel en niet stabiel $\Leftrightarrow$ $\left\{\begin{array}{l}\text{A reducibel en niet nil-}\\ \text{potent.}\end{array}\right.$

(c) x niet semistabiel $\Leftrightarrow$ A nilpotent.

[*] D. Mumford: "Geometric Invariant Theory", Ergeb.Math., Bd.34; Springer Verlag, 1965.

5. Zij R een commutatieve ring met eenheidselement waarin 2 inverteerbaar is.

Zij $S = \{f \in R[X_1,\ldots,X_n,Y_1,\ldots,Y_n] \mid f(X_1,\ldots,X_n,Y_1,\ldots,Y_n) = f(Y_1,\ldots,Y_n,X_1,\ldots,X_n)\}$.

Dan is S een R-algebra van eindig type die wordt voortgebracht door X_i+Y_i $(1 \leq i \leq n)$ en $(X_i-Y_i)(X_j-Y_j)$ $(1 \leq i \leq j \leq n)$.

6. Het grove moduli probleem dat in dit proefschrift wordt behandeld in §1 van hoofdstuk VI kan ook opgelost worden met behulp van de volgende reduktiestelling:

Zij F : Var $\to$ Ens een contravariante functor, X een varieteit en $\Phi : F \to h_X$ een functormorfisme zodat $\Phi(\mathrm{Spec}\ k)$ bijectief is. Zij $\{X_i\}_{i\in I}$ een open overdekking van X en $\alpha_i : X_i \hookrightarrow X$ het inclusiemorfisme. Voor elke $i \in I$ zij F_i : Var $\to$ Ens een deelfunctor van F en $\Phi_i : F_i \to h_{X_i}$ een functormorfisme zó dat onderstaand diagram cartesisch is voor elke varieteit S.

$$\begin{tikzcd} F_i(S) \arrow[r, "\Phi_i(S)"'] \arrow[d, hook] & \mathrm{Hom}(S,X_i) \arrow[d, "\alpha_{i*}"] \\ F(S) \arrow[r, "\Phi(S)"'] & \mathrm{Hom}(S,X) \end{tikzcd}$$

Als voor elke $i \in I$ het paar (X_i,Φ_i) een grove moduliruimte is voor F_i, dan is het paar (X,Φ) een grove moduliruimte voor F.

7. Zij K een lichaam van karakteristiek nul en V een n-dimensionale K-vectorruimte. Procesi identificeert in [*] de vectorruimten $V^* \otimes V$ en End V door: Als $\phi \in V^*$ en $v \in V$ dan is $\phi \otimes v$ het endomorfisme van V gedefinieerd door $(\phi \otimes v)(u) = \langle\phi,u\rangle v$ $(u \in V)$.

In afwijking van de in [*] (p.311) gegeven formule wordt de compositie van endomorfismen onder deze identificatie:

$\phi \otimes v \,.\, \psi \otimes u = \langle\phi,u\rangle\psi \otimes v$.

In het vervolg van het artikel moeten dientengevolge enige correcties worden aangebracht. Zo zal bijvoorbeeld Theorem 1.2 (p.312) moeten luiden:

Zij $\sigma = (i_1 i_2 \ldots i_k)(j_1 j_2 \ldots j_n)\ldots(t_1 t_2 \ldots t_e)$ een ontbinding van $\sigma \in S_i$ in disjuncte cykels (inclusief die van lengte 1), dan geldt voor alle $A_1, A_2, \ldots, A_i \in (K)_n$:

$$\mu_{\sigma^{-1}}(A_1 \otimes \ldots \otimes A_i) = \mathrm{tr}(A_{i_1} A_{i_2} \ldots A_{i_k})\mathrm{tr}(A_{j_1} A_{j_2} \ldots A_{j_n})\ldots\mathrm{tr}(A_{t_1} A_{t_2} \ldots A_{t_e}).$$

[*] C. Procesi: "The Invariant Theory of n×n Matrices", Advances in Math. 19 (1976), 306-381.

8. Definieer T_n, $\mathcal{Q}$ en F zoals in I, §1 van dit proefschrift. Dan geldt $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ waarbij $\mathcal{Q}_1$ het ideaal in T_n is dat voortgebracht wordt door de elementen $F(N_1, N_2, N_3)$ met N_i een monoom in $X_1, X_2, \ldots, X_n$ van graad ≤ 2 en $\mathcal{Q}_2$ door de elementen $F(X_i X_j, X_k, M)$ met $i,j,k \in \{1,\ldots,n\}$ en M een monoom van graad ≥ 3.

9. In een democratische samenleving kan een advocaat ook in loondienst van de overheid zijn praktijk met de vereiste onafhankelijkheid uitoefenen.

10. De in veel natuurbaden bij herhaling omgeroepen woorden "er wordt niet met zand gegooid" doen minder ernstige twijfel rijzen aan het waarnemingsvermogen van de betrokken toezichthouders dan aan de gangbare linguïstische beschrijvingen van de imperatief.

11. Het merkwaardige van merkwaardige produkten is onder andere dat ze merkwaardig heten.